Ph 12c

Homework Assignment No. 8
Due: Thursday, 2 June 2016, 8pm

1. Hyperscaling

The correlation length r_{corr} of a nearly critical system is the characteristic length scale of the critical fluctuations, and the critical exponent ν characterizes how the correlation length diverges as ϵ approaches zero for $\lambda = 0$:

$$r_{\mathrm{corr}} \sim |\epsilon|^{-\nu}. \tag{1}$$

According to the *hyperscaling* hypothesis, the Gibbs free energy close to the critical point behaves like

$$G(\epsilon, \lambda) \sim \left(\frac{L}{r_{\mathrm{corr}}}\right)^{d} \tag{2}$$

for a system in d spatial dimensions, where L is the system's *linear* size. In other words, the Gibbs free energy, an extensive quantity, is proportional to the number of "correlation volumes." The idea behind this hypothesis is that the physics of a single correlation volume remains invariant as $\epsilon \to 0$, but disjoint correlation volumes are nearly independent and hence contribute additively to the free energy. Another way to justify hyperscaling is to assert that, close to the critical point, r_{corr} is the only relevant length scale other than L. Actually, hyperscaling works for some phase second-order transitions but not all — in particular, it fails in Landau theory.

Assuming both the scaling hypothesis and the hyperscaling hypothesis eq.(2), find a relation between the critical exponents ν and α. (This is called *Josephson's identity.*) See eq.(4) and eq. (13) of Homework No. 7 for the statement of the scaling hypothesis and the definition of the exponent α.

2. Evaporative cooling

For a classical ideal gas with N particles at temperature τ, suppose that all particles with kinetic energy greater than τ escape from the container, while all particles with kinetic energy less than τ are retained.

(a) How many particles escape?

(b) What fraction of the total energy escapes?

(c) After a while the retained particles return to thermal equilibrium, but with a reduced energy per particle and hence a reduced temperature τ'. What is τ'?

For parts (a) and (b) you may encounter some definite integrals, which can be evaluated numerically (e.g. with *Mathematica*).

3. Einstein relation on a lattice

In class we derived the diffusion equation by treating diffusion as a random walk on a one-dimensional lattice. Lattice sites are spaced distance Δ apart, and in each time interval ϵ the particle hops either left or right to a neighboring site. If the walk is unbiased (left and right steps are equally probable), then the probability distribution $P(x,t)$ for the particle's position x at time t obeys

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2},$$

where

$$D = \frac{\Delta^2}{2\epsilon}.$$

If a force F is applied that pushes the particles to the right, the walk becomes biased; the particle is more likely to step right than step left. In thermal equilibrium the expectation value of the particle's position moves right at a constant drift velocity.

(a) Recall from Problem 2 in homework set 2 that in thermal equilibrium the rate $\Gamma(1 \to 2)$ for a transition from state 1 to state 2 is related to the rate $\Gamma(2 \to 1)$ for the inverse process according to

$$\frac{\Gamma(1 \to 2)}{\Gamma(2 \to 1)} = e^{(E_1 - E_2)/\tau}.$$

Noting that work $F\Delta$ is required to move the particle one step to the left, find the difference $P_R - P_L$ between the probability the particle takes a step right and the probability it takes a step left, in thermal equilibrium at temperature τ.

(b) The walk is only slightly biased if $F\Delta/\tau \ll 1$. In this limit, find the expectation value for the net distance right traveled by the particle after n steps. Thus derive the *Einstein relation* relating the particle's drift velocity in thermal equilibrium to F, τ, and the diffusion constant D.

4. Einstein relation from linear response theory

Suppose that the Hamiltonian $H^{(0)}$ can be simultaneously diagonalized along with two operators A and B. That is, there is a basis of quantum states $\{|i\rangle\}$ such that each $|i\rangle$ is an eigenstate of H, A, and B with eigenvalues $E_i^{(0)}$, A_i, and B_i respectively. In classical statistical mechanics all operators commute, so this assumption entails no loss of generality.

(a) Consider perturbing the Hamiltonian $H^{(0)}$, replacing it by

$$H^{(\lambda)} = H^{(0)} - \lambda A,$$

where λ is small. We denote by $\langle B\rangle_0$ the expectation value of B in the Boltzmann distribution determined by $H^{(0)}$ at temperature τ and by $\langle B\rangle_\lambda$ the expectation value of B in the Boltzmann distribution determined by $H^{(\lambda)}$ at temperature τ. Expanding to linear order in λ, show that

$$\langle B\rangle_\lambda - \langle B\rangle_0 = \frac{\lambda}{\tau}\left(\langle AB\rangle_0 - \langle A\rangle_0\langle B\rangle_0\right) + O(\lambda^2).$$

(b) For a particle moving diffusively in one dimension, introduce a force F pointing to the right, hence adding the term $-Fx$ to the Hamiltonian where x is the particle position. Using classical statistical mechanics, relate the drift velocity $\langle v\rangle_F$ of the particle in thermal equilibrium to the expectation value $\langle xv\rangle_0$, which expresses how the position and velocity of the particle are correlated in the absence of the force.

To derive the Einstein relation, we should relate $\langle xv\rangle_0$ to the diffusion constant D. Referring to our one-dimensional lattice model, we note that $\langle (x(t+\epsilon) - x(t))^2\rangle = \Delta^2$, and hence

$$\begin{aligned}&\langle v(t+\epsilon/2)\ x(t+\epsilon)\rangle - \langle v(t+\epsilon/2)\ x(t)\rangle \\ &= \left\langle \frac{x(t+\epsilon)-x(t)}{\epsilon}\ x(t+\epsilon)\right\rangle - \left\langle \frac{x(t+\epsilon)-x(t)}{\epsilon}\ x(t)\right\rangle = \frac{\Delta^2}{\epsilon} = 2D.\end{aligned}$$

We obtain the Einstein relation if we assume that $\langle xv\rangle_0$ may be interpreted as

$$\lim_{\delta\to 0^+}\langle v(t)x(t+\delta)\rangle_0 = -\lim_{\delta\to 0^-}\langle v(t)x(t+\delta)\rangle_0 = D.$$

That is, the position is positively correlated with the velocity at a slightly earlier time and negatively correlated with the velocity at a slightly later time. Hence $\langle xv \rangle_0$ should be defined carefully by taking a suitable limit, and its sign depends on how this limit is taken. Resolving this ambiguity properly requires a more detailed treatment of time correlations in thermal equilibrium, which goes beyond the scope of this problem.

5. The cost of erasure

A *bit* is a physical system that can be in either one of two possible states, which we may call 0 and 1. *Erasure* of a bit is a process in which the state of the bit is reset to 0. Erasure is *irreversible* — knowing only the final state 0 after erasure, we cannot determine whether the initial state before erasure was 0 or 1.

This irreversibility implies that erasure incurs an unavoidable thermodynamic cost. According to *Landauer's Principle*, erasing a bit at temperature τ requires work $W \geq \tau \ln 2$. In this problem you will check that a particular procedure for achieving erasure adheres to Landauer's Principle.

Suppose that the two states of the bit both have zero energy. We erase the bit in two steps. In the first step, we bring the bit into contact with a reservoir at temperature $\tau > 0$, and wait for the bit to come to thermal equilibrium with the reservoir. In this step the bit "forgets" its initial value, but the bit is not yet erased because it has not been reset.

We reset the bit in the second step, by slowly turning on a control field λ which splits the degeneracy of the two states. For $\lambda \geq 0$, the state 0 has energy $E_0 = 0$ and the state 1 has energy $E_1 = \lambda$. After the bit thermalizes in step one, the value of λ increases gradually from the initial value $\lambda = 0$ to the final value $\lambda = \infty$; the increase in λ is slow enough that the qubit remains in thermal equilibrium with the reservoir at all times. As λ increases, the probability $P(0)$ that the qubit is in the state 0 approaches unity — *i.e.*, the bit is reset to the state 0, which has zero energy.

(a) For $\lambda \neq 0$, find the probability $P(0)$ that the qubit is in the state 0 and the probability $P(1)$ that the qubit is in the state 1.

(b) How much work is required to increase the control field from λ to $\lambda + d\lambda$?

(c) How much work is expended as λ increases slowly from $\lambda = 0$ to $\lambda = \infty$? (You will have to evaluate an integral, which can be done analytically.)