Physics 12c, Problem Set 1 Solutions

April 6, 2016

[KK 2.3] Quantum harmonic oscillator.

(a) The multiplicity function of a system of N harmonic oscillators of frequency ω with quantum number n is given by Eq. (1.55) of Kittel&Kroemer:

$$g(N,n) = \frac{(N+n-1)!}{n!(N-1)!} \approx \frac{(N+n)!}{n!N!}$$

where we used $N \gg 1$. The entropy of the system is the logarithm of the expression above. Using Sterling formula we can write

$$\sigma(N,n) \approx \log\left(\frac{(N+n)!}{N!n!}\right) \approx (N+n)\log(N+n) - n\log n - N\log N.$$

(b) In order to find the equilibrium temperature, we need an expression of the entropy $\sigma(N,U)$ as a function of the energy, not the quantum number n. The total energy of the system is $U = n\hbar\omega$. Therefore,

$$\sigma(N,U) = \frac{1}{\hbar\omega}\Big[(N\hbar\omega + U)\log(N\hbar\omega + U) - U\log U - N\hbar\omega\log N\Big].$$

Now, we are ready to differentiate $\sigma(N,U)$ to find the equilibrium temperature

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_N = \frac{1}{\hbar\omega}\Big[\log(N\hbar\omega + U) - \log U\Big] = \frac{1}{\hbar\omega}\log(N\hbar\omega/U + 1).$$

This gives

$$U = \frac{N\hbar\omega}{e^{\hbar\omega/\tau} - 1}.$$

[1] The moment-generating function and the central limit theorem.

(a) From the definition, we have

$$\bar{X}(t) = \int_{-\infty}^{\infty} dx q(x) e^{tx} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx e^{tx - x^2/2\sigma^2} = \frac{e^{t^2\sigma^2/2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx' e^{-x'^2/2\sigma^2}$$

where we have defined $x' = x - t$. Then

$$\bar{X}(t) = \frac{e^{t^2\sigma^2/2}}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2} = e^{t^2\sigma^2/2}.$$

(b) We expand $\bar{X}$ in power series:

$$\bar{X}(t) = e^{t^2\sigma^2/2} = \sum_{n=0}^{\infty} \frac{t^{2n}\sigma^{2n}}{2^n n!}.$$

Comparing this expression with

$$\bar{X}(t) = \sum_{n=0}^{\infty} \frac{\langle x^n \rangle t^n}{n!}$$

gives $\langle x^{2n+1} \rangle = 0$ and

$$\langle x^{2n} \rangle = (2n)! \times \frac{\sigma^{2n}}{2^n n!} = \sigma^{2n}(2n-1)!!$$

We use the double factorial notation:

$$(2n-1)!! = \prod_{i=1}^{n} (2i-1) = \frac{(2n)!}{2^n (n)!}$$

(c) As all the x_i's are independent, the integrand factorizes:

$$\bar{U}_N(t) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left[dx p(x_i) e^{tx_i/\sqrt{N}} \right] = \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx p(x_i) e^{tx_i/\sqrt{N}} = \langle e^{tx/\sqrt{N}} \rangle^N = \left[\bar{X}\left(\frac{t}{\sqrt{N}} \right) \right]^N$$

(d) Since

$$\langle e^{tx/\sqrt{N}} \rangle = 1 + \frac{t}{2N} X_1 + \frac{t^2}{2N} X_2 + O(N^{-3/2}),$$

with $X_1 = 0$ we have

$$\bar{U}_N(t) = \left[1 + \frac{t^2}{2N} X_2 + O(N^{-3/2}) \right]^N.$$

In the limit $N \to \infty$, using the indentity

$$\lim_{N\to\infty} \left(1 + \frac{x}{N} \right)^N = e^x,$$

we get

$$\lim_{N\to\infty} \bar{U}_N(t) = e^{t^2 X_2/2},$$

which is the moment-generating function of a Gaussian distribution with $\sigma^2 = X_2$.

The fact that σ is independent of N is because of the normalizing factor $\frac{1}{\sqrt{N}}$.

[2] Biased coin.

In the large N limit, we can treat the function $\ln p(n)$ as "continuous". We have:

$$\sigma_p^{-2} \approx -\Big(\frac{\partial}{\partial n}\Big)^2 \ln p(n) \approx -\Big[\ln p(n+1) - \ln p(n) - \ln p(n) - \ln p(n-1)\Big]$$
$$= \ln \frac{p(n)^2}{p(n+1)p(n-1)} = \ln \frac{\binom{N}{n}^2}{\binom{N}{n+1}\binom{N}{n-1}} = \ln \frac{(n+1)(N-n+1)}{n(N-n)}$$
$$= \ln(1+\frac{1}{n}) + \ln(1+\frac{1}{N-n}) \approx \frac{1}{n} + \frac{1}{N-n} = \frac{N}{n(N-n)}.$$

This expression should be evaluated at $n = pN$, which gives

$$\sigma_p^2 = \frac{n(N-n)}{N} = Np(1-p).$$

For $p = \frac{1}{2}$, we have $\sigma_p^2 = \frac{N}{4}$.

[3] Probability of a large deviation

(a) Define $x' = x/\sigma$, $\alpha = t/\sigma$, then

$$P(x \geq t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_t^\infty dx e^{-x^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty dx' e^{-x'^2/2}.$$

$$P(x \geq t) = \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty dx' \left(-x' e^{-x'^2/2}\right)\left(\frac{-1}{x'}\right)$$

note: $$-xe^{-x^2/2} = \frac{d}{dx} e^{-x^2/2}$$

Integration by parts gives:

$$= \frac{-1}{\sqrt{2\pi}} \left[\frac{e^{-x^2/2}}{x}\right]_\alpha^\infty + \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty dx' \left(-x' e^{-x'^2/2}\right)\left(\frac{1}{x'}\right)^3$$
$$= \frac{-1}{\sqrt{2\pi}} \left[\frac{e^{-x^2/2}}{x}\right]_\alpha^\infty + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-x^2/2}}{x^3}\right]_\alpha^\infty + \mathcal{O}(\alpha^{-5})$$
$$= \frac{e^{-\alpha^2/2}}{\alpha\sqrt{2\pi}} \left(1 - \alpha^{-2} + O(\alpha^{-4})\right)$$
$$= \sqrt{\frac{\sigma^2}{2\pi t^2}} e^{-t^2/2\sigma^2} \left(1 - \frac{\sigma^2}{t^2} + O\Big(\frac{\sigma^4}{t^4}\Big)\right).$$

Therefore $A = B = 1$.

(b) $\alpha = 10$ gives

$$P((x \geq 10\sigma)) \approx \frac{e^{-50}}{10\sqrt{2\pi}} \sim 10^{-24}.$$