Physics 12c - Problem Set 3 - Solutions

April 22, 2016

[1] Surface temperature of Venus.

(a) The power emitted by the sun is:

$$P_\odot = A_\odot \sigma T_\odot^4,$$

where $A_\odot$ is the surface area of the sun and the $\sigma T_\odot^4$ factor follows from the Stefan-Boltzmann law.
The flux of solar radiation at the orbit of Venus is:

$$J_\odot = \frac{4\pi R_\odot^2 \sigma T_\odot^4}{4\pi D^2} = \sigma T_\odot^4 \frac{R_\odot^2}{D^2}.$$

¿From the equilibrium condition (absorbed power = emitted power),

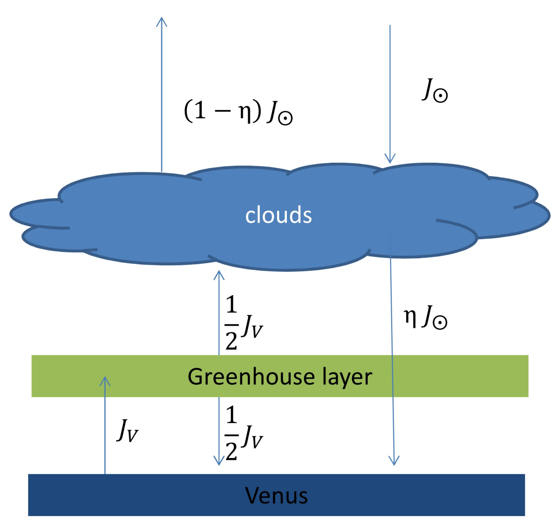
$$\pi R_V^2 J_\odot = 4\pi R_V^2 \sigma T_V^4$$

we have:

$$T_V = T_\odot \sqrt{\frac{R_\odot}{2D}}.$$

Note that only a disc of area πR_V^2 absorbs radiation, whereas the whole surface area of Venus radiates thermally.

(b) The Greenhouse effect is modeled according to the following picture:



¿From the equilibrium condition at the surface of Venus, we find:

$$\eta\pi R_V^2 J_\odot + \frac{1}{2}4\pi R_V^2 J_V = 4\pi R_V^2 J_V$$

which gives:

$$T_V^4 = \frac{\eta}{2\sigma} J_\odot = \frac{\eta}{2}\left(\frac{R_\odot}{D}\right)^2 T_\odot^4.$$

(c) ¿From the equilibrium condition, the emission flux of the cloud is given by

$$J_c = \frac{\pi R_V^2 \eta J_\odot}{4\pi R_V^2} = \frac{\eta}{4} J_\odot.$$

The temperature of the outermost shell is not changed by the presence of additional shells and the emission flux to either side is still J_c. Its energy solely comes from the shell directly below it, which must give out heat at rate $2J_c$. In order to maintain the equilibrium of the second layer, the third must has a emission flux of

$$J_3 = 2 \times 2J_c - J_c = 3J_c.$$

It is easy to prove that for the ith layer, the emission flux is iJ_c. Because for the $i-1$th layer, thermal balance requires:

$$J_i = 2J_{i-1} - J_{i-2} = [2(i-1) - (i-2)]J_c = iJ_c.$$

The ground must sustain the Nth layer, so $J_V = (N+1)J_c$. Then we have:

$$T_V = T_\odot \sqrt[4]{\frac{(N+1)\eta R_\odot^2}{4D^2}}.$$

(d) Bringing in the numbers gives $N \sim 113$. The atmosphere of Venus has high "optical depth".

[2] Debye theory of capillary waves.

Using Debye's method for 2D, we have

$$U \approx \frac{1}{4}\Big(\frac{L}{\pi}\Big)^2 \int_0^\infty dk \frac{2\pi k\hbar\omega}{e^{\hbar\omega/\tau}-1} = \frac{A}{2\pi}\int_0^\infty dk \frac{\hbar\sqrt{C}k^{5/2}}{e^{\hbar\sqrt{Ck^{3/2}/\tau}}-1} = \frac{L^2\tau^{7/3}}{3\pi\hbar^{4/3}C^{2/3}}\int_0^\infty dx \frac{x^{4/3}}{e^x-1}.$$

Using the value of the integral gives

$$U/A \approx 0.1788\tau^{7/3}\hbar^{-4/3}C^{-2/3}.$$

So $\alpha \approx 0.1788$, $\beta = -\frac{2}{3}$, $\gamma = -\frac{4}{3}$ and $\delta = \frac{7}{3}$. Dimensional analysis yields $[U/A] = J/m^2$.

[3] Quantum noise and thermal noise in a harmonic oscillator.

(a) We have

$$\begin{aligned}
\langle n|x(t)x(0)|n\rangle &= \frac{\hbar}{2m\omega}\langle n|(ae^{-i\omega t} + a^\dagger e^{i\omega t})(a+a^\dagger)|n\rangle \\
&= \frac{\hbar}{2m\omega}\langle n|aa^\dagger e^{-i\omega t} + a^\dagger a e^{i\omega t}|n\rangle \\
&= \frac{\hbar}{2m\omega}\Big[(1+n)e^{-i\omega t} + ne^{i\omega t}\Big].
\end{aligned}$$

(b) The partition function is

$$Z(\alpha) = \sum_{n=0}^{\infty} e^{n\alpha} = \frac{1}{1-e^{\alpha}},$$

where $\alpha = -\frac{\hbar\omega}{\tau}$ is a dimensionless parameter. ¿From definition

$$\begin{aligned}\Delta_\tau(t) &= \frac{1}{Z}\sum_{n=0}^{\infty} \frac{\hbar}{2m\omega}\Big[(1+n)e^{-i\omega t} + ne^{i\omega t}\Big]e^{n\alpha} \\ &= \frac{\hbar}{2m\omega Z}\Big[(1+\frac{d}{d\alpha})e^{-i\omega t} + \frac{d}{d\alpha}e^{i\omega t}\Big]Z.\end{aligned}$$

Since

$$\frac{d}{Zd\alpha}Z = \frac{e^{\alpha}}{1-e^{\alpha}} = \frac{1}{e^{-\alpha}-1},$$

we have

$$\begin{aligned}N_\tau(\omega) &= \frac{\hbar}{2m\omega}\frac{1}{e^{-\alpha}-1} = \frac{\hbar}{2m\omega}\frac{1}{e^{\hbar\omega/\tau}-1} \\ P_\tau(\omega) &= \frac{\hbar}{2m\omega}\Big[1+N_\tau(\omega)\Big] = \frac{\hbar}{2m\omega}\frac{e^{\hbar\omega/\tau}}{e^{\hbar\omega/\tau}-1}.\end{aligned}$$

Therefore

$$\frac{N_\tau(\omega)}{P_\tau(\omega)} = e^{-\hbar\omega/\tau}.$$

(c) The limit of $\tau \to 0$ corresponds to $\alpha \to -\infty$, and Z=1. $\frac{d}{d\alpha}Z = 0$ gives

$$\lim_{\tau\to 0}\Delta_\tau(t) = \frac{\hbar}{2m\omega}e^{-i\omega t}.$$

The limit of $\tau \to \infty$ corresponds to $\alpha \to 0^-$. Then

$$\frac{d}{Zd\alpha}Z = \frac{1}{e^{-\alpha}-1} = -\alpha^{-1} = \frac{\tau}{\hbar\omega} \gg 1.$$

Therefore,

$$\lim_{\tau\to\infty}\Delta_\tau(t) = \frac{\tau}{\hbar\omega}\frac{\hbar}{2m\omega}(e^{-i\omega t}+e^{i\omega t}) = \frac{\tau}{m\omega^2}\cos(\omega t),$$

which is independent of $\hbar$. This is guaranteed because for high temperature, $[a, a^\dagger] \ll \langle n \rangle$, and all operators are effectively c-numbers.

4. Lifetime of a black hole

A black hole with mass M (and hence energy Mc^2) has surface area $A = 4\pi R^2$, where $R = 2GM/c^2$ is its "Schwarzschild radius." Its entropy is

$$\sigma = \frac{A}{4L_{\rm Pl}^2},$$

where

$$L_{\rm Pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6\times 10^{-35} m,$$

is the "Planck length" (G is Newton's gravitational constant).

(a) *Find the temperature* τ *of a black hole, expressed in terms of* M, G, $\hbar$, *and* c.

We'll determine the temperature by expressing the entropy σ as a function of the mass M and using the identity $d\sigma/dE = 1/\tau$. We have

$$\sigma = \frac{A}{4L_{\text{Pl}}^2} = \left(\frac{c^3}{4\hbar G}\right) 4\pi \left(\frac{2GM}{c^2}\right)^2 = \frac{4\pi GM^2}{\hbar c};$$

therefore,

$$\frac{1}{\tau} = \frac{d\sigma}{dE} = \frac{1}{c^2}\frac{d\sigma}{dM} = \frac{8\pi GM}{\hbar c^3},$$

and hence

$$\tau = \frac{\hbar c^3}{8\pi GM}.$$

(b) *Black holes evaporate. Assuming the black hole radiates like a black body with temperature* τ *and surface area* A, *show that its mass* $M(t)$ *decreases as a function of time according to*

$$M^2 \frac{dM}{dt} = -B$$

where B *is a constant. Express* B *in terms of* G, $\hbar$, *and* c.

The rate at which the black hole loses energy is the rate at which it emits black body radiation; therefore,

$$\frac{dE}{dt} = c^2 \frac{dM}{dt} = -\sigma_B \left(4\pi R^2\right) \tau^4 = -4\pi\sigma_B \left(\frac{2GM}{c^2}\right)^2 \left(\frac{\hbar c^3}{8\pi GM}\right)^4 = -\frac{16\pi\sigma_B \hbar^4 c^8}{(8\pi)^4 (GM)^2}$$

and from $\sigma_B = \frac{\pi^2}{60\hbar^3 c^2}$ we obtain

$$\frac{dM}{dt} = -\frac{\hbar c^4}{15360\pi (GM)^2},$$

which implies

$$-M^2 \frac{dM}{dt} \equiv B = -\frac{\hbar c^4}{15360\pi G^2}.$$

(c) *By solving this differential equation, find the time* t_M *for a black hole to evaporate completely if its initial mass is* M. *Express* t_M *in terms of* B *and* M.

Integrating we find

$$\int_0^t dt\ M^2 \frac{dM}{dt} = \frac{1}{3}\left(M^3(t) - M^3(0)\right) = -Bt,$$

or

$$M(t) = \left(M(0)^3 - 3Bt\right)^{1/3};$$

therefore, $M(t) = 0$ at time

$$t_M = \frac{M^3}{3B}.$$

where $M = M(0)$ is the initial mass.

(d) *What is the lifetime of a solar mass black hole? (Don't be surprised if it's a long time.)*

Plugging in B from (b), we obtain

$$t_M = 5120\pi \left(\frac{\hbar G}{c^5}\right)^{1/2} \left(\frac{G}{\hbar c}\right)^{3/2} M^3 \equiv 5120\pi \; t_{\rm Pl} \; \left(\frac{M}{M_{\rm Pl}}\right)^3 ,$$

where

$$t_{\rm Pl} \equiv \left(\frac{\hbar G}{c^5}\right)^{1/2} \approx 5.4 \times 10^{-44} \text{ s}$$

is the "Planck time," and

$$M_{\rm Pl} \equiv \left(\frac{\hbar c}{G}\right)^{1/2} \approx 2.2 \times 10^{-8} \text{ kg}$$

is the "Planck mass." Using $M_\odot \approx 2.0 \times 10^{30}$ kg. we obtain

$$t_M \approx 6.5 \times 10^{74} \text{ s} \approx 2.1 \times 10^{67} \text{ yr}.$$