Physics 12c, Problem Set 4 Solutions

April 29, 2016

Ascent of sap in trees (Pr. 5.12 of Kittel/Kroemer)

The chemical potential must be the same at the top ($h = H$) and bottom ($h = 0$) of the tree. It is given by

$$\mu = \tau \log(n/n_Q) + Mgh. \tag{0.1}$$

We can write $n = r(h)n_0$, where n_0 is the vapor pressure of air at the roots, and r is a function of height such that $r(0) = 1$ and $r(H) = 0.9$. Then the the chemical potential becomes

$$\mu = \tau \log(rn_0/n_Q) + Mgh = \tau \log r + Mgh + const, \tag{0.2}$$

where we used the fact that $\tau \log(n_0/n_Q)$ is a constant independent of height (assuming the temperature does not depend on height).

Thus, the equality of chemical potentials at $h = 0$ and $h = H$ implies

$$\tau \log r(H) + MgH = \tau \log r(0) = \tau \log 1 = 0, \tag{0.3}$$

so that

$$H = -\frac{\tau \log r(H)}{Mg}. \tag{0.4}$$

For $r(H) = 0.9$, this evaluates to H being about 1500 meters. This is greater than the height of any trees on Earth, so trees can transport water to the leaves without expending energy.

[1] Isentropic model of the atmosphere

(a) The consider a slab of gas with volume $A \times dz$. The gravitational force is

$$F_g = \rho dV g = (\rho A dz)g \tag{1.1}$$

where ρ is density. The force due to the gradient of pressure is

$$F_b = -Adp \tag{1.2}$$

At equilibrium,

$$\begin{aligned} F_b &= F_g \\ \frac{dp}{dz} = -\rho g = -\frac{mNg}{V} &= -\frac{mgp}{\tau} \end{aligned} \tag{1.3}$$

(b)

$$\frac{d\tau}{dz} = \frac{d\tau}{d\rho}\frac{dp}{dz} = \frac{n - p\frac{dn}{dp}}{n^2}\frac{dp}{dz}. \tag{1.4}$$

Since $pn^{-\gamma}$ is constant.

$$\begin{aligned}\frac{d(pn^{-\gamma})}{dp} &= n^{-\gamma} - p\gamma n^{-(\gamma+1)}\frac{dn}{dp} = 0 \\ \frac{dn}{dp} &= \frac{n}{\gamma p}. \end{aligned} \tag{1.5}$$

So,

$$\begin{aligned}\frac{d\tau}{dz} &= -\frac{1}{n}(1 - \frac{1}{\gamma})(\frac{mgp}{\tau}) \\ &= -mg(1 - \frac{1}{\gamma}) \end{aligned} \tag{1.6}$$

(c)

$$\frac{d\tau}{dz} = -9.4K/km \tag{1.7}$$

[2] Fluctuations in particle number

(a)

$$\begin{aligned}\langle N\rangle &= \frac{1}{\mathcal{Z}}\sum_s N_s e^{(N_s\mu - \epsilon_s)/\tau} \\ \frac{\partial\langle N\rangle}{\partial\mu}\Big|_\tau &= \frac{1}{\mathcal{Z}}\frac{1}{\tau}\sum_s N_s^2 e^{(N_s\mu-\epsilon_s)/\tau} - \frac{1}{\mathcal{Z}^2}\frac{\partial\mathcal{Z}}{\partial\mu}\sum_s N_s e^{(N_s\mu-\epsilon_s)/\tau} \\ &= \frac{1}{\tau}\langle N^2\rangle - \frac{1}{\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial\mu}\langle N\rangle \\ &= \frac{1}{\tau}\langle N^2\rangle - \frac{1}{\tau}\langle N\rangle^2 \end{aligned} \tag{2.1}$$

So,

$$\tau\frac{\partial\langle N\rangle}{\partial\mu}\Big|_\tau = \langle N^2\rangle - \langle N\rangle^2 = \langle(N - \langle N\rangle)^2\rangle \tag{2.2}$$

(b) For classical ideal gas,

$$\langle N\rangle = \sum_s \exp[(\mu - \epsilon_s)/\tau] \tag{2.3}$$

So,

$$\tau\frac{\partial\langle N\rangle}{\partial\mu}\Big|_\tau = \langle N\rangle. \tag{2.4}$$

¿From (a),

$$\begin{aligned}\langle N\rangle &= \langle(\Delta N)^2\rangle \\ \frac{\langle(\Delta N)^2\rangle}{\langle N\rangle^2} &= \frac{1}{\langle N\rangle} \end{aligned} \tag{2.5}$$

Hence, for $N \gg 1$, the fluctuations in particle number are small.

[3] Ideal gas adsorption

Using the distribution function $f(\epsilon) = e^{(\mu-\epsilon)/\tau}$*, the particle number* N *of a classical ideal gas in a box with volume* V *can be expressed as*

$$N = e^{\mu/\tau} \sum_a e^{-\epsilon_a/\tau} = e^{\mu/\tau} \left(n_Q V\right),$$

where the sum is over all single-particle orbitals, and $n_Q = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$ *is the quantum concentration. Hence the chemical potential is*

$$\mu = \tau \ln\left(n/n_Q\right),$$

where n *is the concentration of particles per unit volume.*

(*a*) *Use the same method to find the chemical potential of a two-dimensional classical ideal gas, expressed in terms of the concentration* $\bar{n}$ *of particles per unit area* A.

Approximating the sum over orbitals by an integral over wave numbers we have

$$N = e^{\mu/\tau} \mathrm{A} \int \frac{dk_x dk_y}{(2\pi)^2} \exp\left(-\frac{(\hbar k_x)^2 + (\hbar k_y)^2}{2m\tau}\right) = e^{\mu/\tau} \mathrm{A} \left(\frac{m\tau}{2\pi\hbar^2}\right). \tag{3.1}$$

Therefore,

$$e^{\mu/\tau} = \frac{\bar{n}}{\bar{n}_Q}, \tag{3.2}$$

where

$$\bar{n} = \frac{N}{\mathrm{A}}, \quad \bar{n}_Q = \frac{m\tau}{2\pi\hbar^2}, \tag{3.3}$$

and hence

$$\mu = \tau \ln\left(\bar{n}/\bar{n}_Q\right). \tag{3.4}$$

Suppose that the walls of a box containing a classical ideal gas with temperature τ *can adsorb the gas particles, where it costs energy* Δ *to remove an adsorbed particle from the wall. The adsorbed particles move freely along the walls, and so can be modeled as an ideal gas confined to two dimensions. The adsorbed particles are in thermal and diffusive equilibrium with the gas particles contained inside the box.*

(*b*) *The concentration* $\bar{n}$ *of adsorbed particles per unit area is related to the concentration* n *of gas particles per unit volume according to*

$$\bar{n} = Cn$$

where C *depends on* Δ*,* τ*, and the particle mass* m*. Find* C*.*

We denote the chemical potential for the three-dimensional gas as μ_3 and for the two-dimensional gas as μ_2. In diffusive equilibrium

$$(\mu_3)_{\mathrm{int}} + (\mu_3)_{\mathrm{ext}} = (\mu_2)_{\mathrm{int}} + (\mu_2)_{\mathrm{ext}}, \tag{3.5}$$

and the external chemical potential difference is

$$(\mu_3)_{\mathrm{ext}} - (\mu_2)_{\mathrm{ext}} = \Delta \tag{3.6}$$

if it costs energy Δ to remove a particle from the two-dimensional gas and add it to the three-dimensional gas. From (a) we have

$$(\mu_2)_{\text{int}} - (\mu_3)_{\text{int}} = \tau \ln(\bar{n}/\bar{n}_Q) - \tau \ln(n/n_Q) = \Delta; \tag{3.7}$$

dividing by τ and exponentiating we obtain

$$\frac{\bar{n}}{\bar{n}_Q} = \frac{n}{n_Q} e^{\Delta/\tau}, \tag{3.8}$$

or

$$\bar{n} = \frac{\bar{n}_Q}{n_Q} e^{\Delta/\tau} n = Cn, \quad \text{where} \quad C = \sqrt{\frac{2\pi\hbar^2}{m\tau}} e^{\Delta/\tau}. \tag{3.9}$$

[4] Diffusive contact

Two identical systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are both in thermal contact with a large reservoir and in diffusive contact with one another. For both systems, the free energy F is related to the particle number N by $F = cN^2$, where c is an N-independent constant (the same constant for both systems).

(a) *A battery maintains a chemical potential difference*

$$\Delta = \mu_{2,\text{ext}} - \mu_{1,\text{ext}} > 0$$

between the two systems. In diffusive equilibrium, find the number N_1 of particles in $\mathcal{S}_1$ and the number N_2 of particles in $\mathcal{S}_2$, expressed in terms of Δ, c, and the total particle number $N = N_1 + N_2$.

Using the free energy, we find

$$F = cN^2 \Rightarrow \mu = \partial F/\partial N = 2cN. \tag{4.1}$$

In diffusive equilibrium,

$$\mu_{1,\text{total}} = \mu_{1,\text{ext}} + 2cN_1 = \mu_{2,\text{total}} = \mu_{2,\text{ext}} + 2cN_2. \tag{4.2}$$

¿From $N_2 = N - N_1$ and $\Delta = \mu_{2,\text{ext}} - \mu_{1,\text{ext}}$ we obtain

$$2cN_1 = 2c(N - N_1) + \Delta \Rightarrow 4cN_1 = 2cN + \Delta; \tag{4.3}$$

hence

$$N_1 = N/2 + \Delta/4c \quad \text{and} \quad N_2 = N/2 - \Delta/4c. \tag{4.4}$$

(b) *Now the battery is disconnected, and useful work is extracted isothermally as the particles flow slowly from $\mathcal{S}_1$ to $\mathcal{S}_2$ until diffusive equilibrium is reestablished. How much work is extracted?*

In a reversible isothermal process, the work done by the system is the decrease in the system's Helmholtz free energy. Before the battery is disconnected the total free energy is

$$F_{\text{initial}} = cN_1^2 + cN_2^2 = c(N/2 + \Delta/4c)^2 + c(N/2 - \Delta/4c)^2 = cN^2/2 + \Delta^2/8c.$$

After the battery is disconnected and diffusive equilibrium is reestablished with $\Delta = 0$, the total free energy is

$$F_{\text{final}} = cN_1^2 + cN_2^2 = 2c(N/2)^2 = cN^2/2. \tag{4.5}$$

The work done W is the change in free energy

$$W = F_{\text{initial}} - F_{\text{final}} = \Delta^2/8c. \tag{4.6}$$