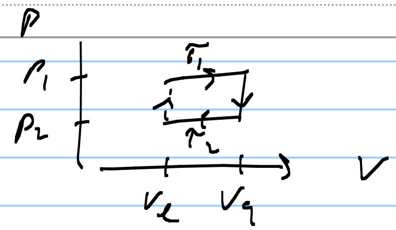
Physics 12c, Problem Set 7 Solutions, May 2016

Note Title 5/27/2016

1a)



work = enclosed area

$= (p_1 - p_2)(V_g - V_\ell)$

b) Heat added at τ_1 is L

$$W = (p_1 - p_2)(V_g - V_\ell) = L\,\frac{\tau_1 - \tau_2}{\tau_1}$$

$$\Longrightarrow \quad \frac{p_1 - p_2}{\tau_1 - \tau_2} = \frac{L}{\tau_1 (V_g - V_\ell)}$$

This agrees with Clausius Clapeyron in the limit $\tau_2 \to \tau_1$.

$$\left(\frac{dp}{d\tau}\right)_{coex} = \frac{L}{\tau \Delta V}$$

It's a derivation of C-C, since cycle is reversible.

c) Power $\quad W = L \eta_c \times \left(\frac{\text{moles}}{\text{sec}}\right), \quad \eta_c = \frac{\tau_1 - \tau_2}{\tau_1}$

$$\frac{\text{Vol}}{\text{sec}} = \frac{\text{moles}}{\text{sec}} \times \frac{m^3}{\text{mole}} = \frac{\text{moles}}{\text{sec}} \times (18 \times 10^{-6})$$

— 18 = molecular wt. of H_2O

$$\frac{\text{height}}{\text{sec}} = \frac{1}{\text{area}}\,\frac{\text{Vol}}{\text{sec}} = \frac{1}{A}(18 \times 10^{-6})\,\frac{W}{L\eta_c}$$

$$\frac{\text{height}}{\text{day}} = (24 \times 3600) \times (18 \times 10^{-6})\,\frac{W}{A L \eta_c}$$

$$= 1.56\,\frac{W}{A L \eta_c}$$

d) $$\eta_c = \frac{\tau_1 - \tau_2}{\tau_1} = \frac{300 - 200}{300} = \frac{1}{3}$$

$$\Longrightarrow \quad \frac{\text{height}}{\text{day}} = (1.56)(3)\,\frac{2 \times 10^{14}\,\text{watt}}{(10^{11} m^2)(41 \times 10^3\,J/\text{mole})} = .22\,\frac{m}{\text{day}}$$

9 inches of rain per day -- sounds about right.

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[2] Scaling hypothesis from Landau theory

From

$$G(\epsilon,\lambda) = [F(\epsilon,\xi) - \lambda\xi]_{\text{stat wrt }\xi}$$

and

$$F(\epsilon,\xi) = \frac{1}{2}A\epsilon\xi^2 + \frac{1}{4}B\xi^4,$$

we infer that

$$\begin{aligned} G(\Omega^p\epsilon, \Omega^q\lambda) &= \left[\frac{1}{2}A\Omega^p\epsilon\xi^2 + \frac{1}{4}B\xi^4 - \Omega^q\lambda\xi\right]_{\text{stat wrt }\xi} \\ &= \Omega\left[\frac{1}{2}A\epsilon\left(\Omega^{(p-1)/2}\xi\right)^2 + \frac{1}{4}B\left(\Omega^{-1/4}\xi\right)^4 - \lambda\left(\Omega^{q-1}\xi\right)\right]_{\text{stat wrt }\xi}. \end{aligned}$$

Now we want to show that the right-hand side becomes $\Omega G(\epsilon,\lambda)$ for an appropriate choice of p and q. In fact if we choose $(p-1)/2 = -1/4$ and $q-1 = -1/4$, then the quantity inside the square brackets becomes a function of $\Omega^{-1/4}\xi$, and we have

$$G(\Omega^p\epsilon, \Omega^q\lambda) = \Omega\left[\frac{1}{2}A\epsilon\left(\Omega^{-1/4}\xi\right)^2 + \frac{1}{4}B\left(\Omega^{-1/4}\xi\right)^4 - \lambda\left(\Omega^{-1/4}\xi\right)\right]_{\text{stat wrt }\xi}.$$

But rescaling ξ by $\Omega^{-1/4}$ does not change the value of the quantity in square brackets at its stationary point, and so we find

$$G(\Omega^p\epsilon, \Omega^q\lambda) = \Omega G(\epsilon,\lambda)$$

when we choose

$$p = 1/2, \quad q = 3/4.$$

[3] Critical exponents from the scaling hypothesis

(a) The order parameter is given by $\xi = -\left(\frac{\partial G}{\partial\lambda}\right)_\tau$. Differentiating the scaling hypothesis with respect to λ gives:

$$\Omega^q\xi(\Omega^p\epsilon, \Omega^q\lambda) = \Omega\xi(\epsilon,\lambda) \tag{S1}$$

We set $\lambda = 0$, and take the limit $\epsilon \to 0^-$ (i.e. approach τ_C from below) while holding $\Omega^p\epsilon$ fixed. This means that $\Omega \propto |\epsilon|^{-1/p} = (-\epsilon)^{-1/p}$ since $\epsilon < 0$ in this case[1]. Furthermore, for $\Omega^p\epsilon$ fixed, $\xi(\Omega^p\epsilon, 0)$ is just a constant. Therefore,

$$\xi \sim \Omega^{q-1} \sim (-\epsilon)^{\frac{1-q}{p}} \qquad \Rightarrow \qquad \beta = \frac{1-q}{p}.$$

[1] We usually want to study the scaling behavior while on the same side of τ_C, hence we can take Ω as a positive quantity.

(b) Set $\epsilon = 0$ in eq. (S1), and take the limit $\lambda \to 0$, while holding $\Omega^q \lambda$ fixed, i.e. $\Omega \propto \lambda^{-1/q}$. Therefore, we get

$$\xi \sim \Omega^{q-1} \sim \lambda^{-\frac{q-1}{q}}$$
$$\Rightarrow \qquad \lambda \sim \xi^{\frac{q}{1-q}} \qquad \Rightarrow \qquad \delta = \frac{q}{1-q}\,.$$

(c) Recall that $\sigma = -\left(\frac{\partial G}{\partial \tau}\right)_\lambda$, so the heat capacity is given by $C_\lambda = -\tau\left(\frac{\partial^2 G}{\partial \tau^2}\right)_\lambda$. Differentiating the scaling hypothesis twice with respect to τ, we get

$$\Omega^{2p} C_\lambda(\Omega^p \epsilon, \Omega^q \lambda) = \Omega C_\lambda(\epsilon, \lambda).$$

Set $\lambda = 0$, and take the limit $\epsilon \to 0$ while holding $\Omega^p \epsilon$ fixed, i.e. $\Omega \propto |\epsilon|^{-1/p}$. Then,

$$C_\lambda \sim \Omega^{2p-1} \sim |\epsilon|^{-\frac{2p-1}{p}} \qquad \Rightarrow \qquad \alpha = 2 - \frac{1}{p}\,.$$

(d) For $p = 1/2$ and $q = 3/4$, we find $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$ as expected.

(e) Using the expressions for β and δ from problem 2, we can write:

$$\delta = \frac{q}{1-q} \qquad \Rightarrow \qquad q = \frac{\delta}{1+\delta},$$
$$\beta = \frac{1-q}{p} = \frac{1-\frac{\delta}{1+\delta}}{p} = \frac{1}{p(1+\delta)} \qquad \Rightarrow \qquad \frac{1}{p} = \beta(1+\delta).$$

Therefore, $\alpha = 2 - \frac{1}{p} = 2 - \beta(1+\delta)$, which is known as the Griffiths relation.

(f) Using the expressions for α and β from problem 2, we can write:

$$\alpha = 2 - \frac{1}{p} \qquad \Rightarrow \qquad \frac{1}{p} = 2 - \alpha,$$
$$\beta = \frac{1-q}{p} = 2 - \alpha - \frac{q}{p} \qquad \Rightarrow \qquad \frac{q}{p} = 2 - \alpha - \beta\,.$$

Therefore, $\gamma = \frac{2q}{p} - \frac{1}{p} = 2(2-\alpha-\beta) - (2-\alpha) = 2 - \alpha - 2\beta$, which is known as the Rushbrooke relation.

[4] Equation of state from the scaling hypothesis

(a) Differentiating both sides of the scaling hypothesis

$$G(\epsilon, \lambda) = \Omega^{-1} G(\Omega^p \epsilon, \Omega^q \lambda)\ ,$$

we find

$$\xi(\epsilon, \lambda) = -\left(\frac{\partial G}{\partial \lambda}\right)_\tau = \Omega^{q-1} \xi(\Omega^p \epsilon, \Omega^q \lambda)\ .$$

Now choose Ω so that $\Omega^p = \epsilon^{-1}$, or $\Omega = \epsilon^{-1/p}$, and we have

$$\xi(\epsilon, \lambda) = \epsilon^{(1-q)/p} \xi\left(1, \epsilon^{-q/p}\lambda\right) \Rightarrow \frac{\xi(\epsilon, \lambda)}{\epsilon^{(1-q)/p}} = \xi\left(1, \frac{\lambda}{\epsilon^{q/p}}\right) = f\left(\frac{\lambda}{\epsilon^{q/p}}\right)\ .$$

Therefore,

$$a = \frac{1-q}{p} = \beta\ , \quad b = \frac{q}{p} = \beta\delta\ .$$

(b) Differentiating we find

$$\lambda = \frac{\partial}{\partial \xi} F(\epsilon, \xi) = A\epsilon\xi + B\xi^3,$$

and therefore

$$\lambda\epsilon^{-b} = A\epsilon^{1-b}\xi + B\epsilon^{-b}\xi^3 = A\left(\epsilon^{1-b}\xi\right) + B\left(\epsilon^{-b/3}\xi\right)^3;$$

This has the form $h(\xi/\epsilon^a)$ if $a = b - 1 = b/3$, which has the solution $b = 3/2$ and $a = 1/2$. The function h is

$$h(x) = Ax + Bx^3.$$

To check: in Landau theory, where $p = 1/2$ and $q = 3/4$, the result from (a) becomes

$$a = \frac{1-q}{p} = \frac{1/4}{1/2} = 1/2, \quad b = \frac{q}{p} = \frac{3/4}{1/2} = 3/2 \ .$$