Physics 12c, Problem Set 8 Solutions

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[1] Hyperscaling

For $\lambda = 0$, the scaling hypothesis becomes

$$G(\Omega^p \epsilon) = \Omega G(\epsilon);$$

hence choosing $\Omega^p \epsilon =$ constant, or $\Omega \sim |\epsilon|^{-1/p}$, we find

$$G(\epsilon) \sim \Omega^{-1} \sim |\epsilon|^{1/p}.$$

The hyperscaling relation, together with the definition of ν implies

$$G(\epsilon) \sim (r_{\text{corr}})^{-d} \sim |\epsilon|^{d\nu}.$$

Comparing the two expressions for $G(\epsilon)$ we find

$$d\nu = 1/p.$$

From Problem 2c, we have $1/p = 2 - \alpha$; therefore

$$d\nu = 2 - \alpha,$$

the Josephson identity.

[2] Evaporative cooling

(a) The Maxwell distribution is of the form

$$P(v) = \sqrt{\frac{2}{\pi}} \left(\frac{m}{\tau}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/\tau}. \tag{2.1}$$

Check the normalization,

$$\begin{aligned} \int_0^\infty dv P(v) &= \sqrt{\frac{2}{\pi}} 2^{3/2} \int_0^\infty dv \left(\frac{m}{2\tau}\right)^{1/2} \frac{1}{2\tau} mv^2 e^{-\frac{1}{2}mv^2/\tau} \\ &= \frac{4}{\sqrt{\pi}} \int_0^\infty dx \left(x^2 e^{-x^2}\right) = 1, \end{aligned} \tag{2.2}$$

where

$$x = \left(\frac{1}{2\tau} mv^2\right). \tag{2.3}$$

Probability that $\frac{1}{2}mv^2 > x_0$ is

$$P(x_0) = \frac{4}{\sqrt{\pi}} \int_{x_0}^\infty dx \left(x^2 e^{-x^2}\right). \tag{2.4}$$

And

$$P(x_0 = 1) = 0.572407, \tag{2.5}$$

which is the fraction of particle that escapes.

(b) Number of particles with speed in $(v, v+dv)$ is $NP(v)dv$, carrying energy $\frac{1}{2}mv^2NP(v)dv$. Total energy is

$$U = N\int_0^\infty dvP(v)\frac{1}{2}mv^2 = N\tau\frac{4}{\sqrt{\pi}}\int_0^\infty dx x^4 e^{-x^2} = \frac{3}{2}N\tau. \tag{2.6}$$

Eenrgy of particles with $\frac{1}{2\tau}mv^2 > x_0$ is

$$U(x_0) = N\tau\frac{4}{\sqrt{\pi}}\int_{x_0}^\infty dx x^4 e^{-x^2} = \frac{3}{2}N\tau g(x_0), \tag{2.7}$$

where $g(x_0)$ is fraction of energy lost.

$$g(1) = 0.849145 \tag{2.8}$$

(c) The remaining particles have energy

$$\bar{U} = (1-g)U = (1-g)\frac{3}{2}N\tau. \tag{2.9}$$

$$\bar{N} = (1-f)N \tag{2.10}$$

After evaporation,

$$\bar{U} = (1-g)\frac{3}{2}N\tau = \frac{3}{2}\bar{N}\bar{\tau} = \frac{3}{2}(1-f)N\bar{\tau}. \tag{2.11}$$

So,

$$\bar{\tau} = \frac{1-g}{1-f}\tau = 0.3528\tau, \tag{2.12}$$

for $x_0 = 1$.

[3] Einstein relation on a lattice

(a)

$$\frac{\Gamma(s \to s+1)}{\Gamma(s+1 \to s)} = e^{F\Delta/\tau}. \tag{3.13}$$

So,

$$\frac{P_R}{P_L} = e^{F\Delta/\tau} \to$$
$$P_R = \frac{e^{F\Delta/\tau}}{1+e^{F\Delta/\tau}}, P_L = \frac{1}{1+e^{F\Delta/\tau}}. \tag{3.14}$$

And,

$$P_R - P_L = \frac{e^{F\Delta/\tau}-1}{e^{F\Delta/\tau}+1} = tanh\left(\frac{F\Delta}{2\tau}\right). \tag{3.15}$$

(b) For $\frac{F\Delta}{\tau} << 1$, $P_R - P_L \sim \frac{F\Delta}{2\tau}$. In n steps, the time $t = n\epsilon$, in expectation. The particle hops distance is

$$n\Delta(P_R - P_L) = n\Delta\left(\frac{F\Delta}{2\tau}\right) = n\epsilon\left(\frac{\Delta^2}{2\epsilon}\right)\frac{F}{\tau} = tD\frac{F}{\tau}. \tag{3.16}$$

So the drift speed is

$$\frac{distance}{time} = D\frac{F}{\tau}. \tag{3.17}$$

Mobility $b = \frac{v_{drift}}{F} = \frac{F}{\tau}$.

[4] Einstein relation from linear response theory

(a)

$$Z_0 = \sum_a e^{-E_a/\tau} \tag{4.18}$$

and

$$Z_\lambda = \sum_a e^{-(\epsilon_a - \lambda A_a)/\tau} \sim Z_0 + \frac{\lambda}{\tau} Z_0 < A > . \tag{4.19}$$

And we know that

$$\sum_a B_a e^{-(\epsilon_a - \lambda A_a)/\tau} \sim Z_0 < B > + \frac{\lambda}{\tau} Z_0 < AB > . \tag{4.20}$$

Therefore,

$$< B >_\lambda \frac{Z_0 < B > + \frac{\lambda}{\tau} Z_0 < AB >}{Z_0 + \frac{\lambda}{\tau} Z_0 < A >} \sim < B >_0 + \frac{\lambda}{\tau} (< AB >_0 - < A >_0 < B >_0). \tag{4.21}$$

(b) Next consider $H \rightarrow H - Fx$. Find drift velocity $< v >_F$, where $< v >_0 = 0$.

$$< v >_F = \frac{F}{\tau} < vx >_0 \tag{4.22}$$

[5] The cost of erasure

(a) According to Boltzmann distribution, the probability is proportional to $e^{-E/\tau}$. Thus, we have

$$P(0) = \frac{1}{1 + e^{-\lambda/\tau}} \tag{5.23}$$

$$P(1) = \frac{e^{-\lambda/\tau}}{1 + e^{-\lambda/\tau}} \tag{5.24}$$

(b)

$$dW = P(0) dE_0 + P(1) dE_1 \tag{5.25}$$

$$dW = \frac{e^{-\lambda/\tau}}{1 + e^{-\lambda/\tau}} d\lambda \tag{5.26}$$

(c)

$$W = \int_0^\infty dW = \int_0^\infty \frac{e^{-\lambda/\tau}}{1 + e^{-\lambda/\tau}} d\lambda = \tau \ln 2 \tag{5.27}$$

The result agrees with the Landauer's Principle.