Physics 12c Midterm Solutions
May 2016

Note Title 4/30/2016

1a) Energy is conserved $\Rightarrow \quad \Delta U_1 + \Delta U_2 = 0.$

$\Delta U = C \Delta \tau \Rightarrow \quad \Delta U_1 = C(\tau - \tau_1) = -\Delta U_2 = -C(\tau - \tau_2).$

Therefore, $\quad 2C\tau = C(\tau_1 + \tau_2) \Rightarrow \boxed{\tau = \frac{1}{2}(\tau_1 + \tau_2)}$

b) $dU = \tau d\sigma \Rightarrow \quad \Delta\sigma = \int dU/\tau = C\int d\tau/\tau.$

Therefore, $\quad \Delta\sigma_1 = C\int_{\tau_1}^{\tau} d\tau/\tau = C\ln(\tau/\tau_1)$

$\Delta\sigma_2 = C\int_{\tau_2}^{\tau} d\tau/\tau = C\ln(\tau/\tau_2)$

$\Delta\sigma_{total} = \Delta\sigma_1 + \Delta\sigma_2 = C\ln(\tau^2/\tau_1\tau_2)$

$$\Rightarrow \boxed{\Delta\sigma_{total} = C\ln\left[\frac{(\tau_1+\tau_2)^2}{4\tau_1\tau_2}\right]}$$

c) $(\tau_1 + \tau_2)^2 = (\tau_1 - \tau_2)^2 + 4\tau_1\tau_2 \Rightarrow \dfrac{(\tau_1+\tau_2)^2}{4\tau_1\tau_2} = 1 + \dfrac{(\tau_1-\tau_2)^2}{4\tau_1\tau_2}$

Thus, $\dfrac{(\tau_1+\tau_2)^2}{4\tau_1\tau_2} \geq 1$ and $\dfrac{(\tau_1+\tau_2)^2}{4\tau_1\tau_2} > 1$ for $\tau_1 \neq \tau_2$

Therefore, $\quad \Delta\sigma_{Total} = C\ln\left[\dfrac{(\tau_1+\tau_2)^2}{4\tau_1\tau_2}\right] > 0.$

2a) $N = V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\varepsilon_k/T} - 1} = \frac{V}{2\pi^2} \int dk\, k^2 \frac{1}{e^x - 1}$

$\Rightarrow \quad 2k\,dk = \frac{2mT}{\hbar^2} dx$ where $x = \hbar^2 k^2 / 2mT$.

$k = \left(\frac{2mT}{\hbar^2}\right)^{\frac{1}{2}} x^{\frac{1}{2}}$

$\Rightarrow \quad N = \frac{V}{4\pi^2} \left(\frac{2mT}{\hbar^2}\right)^{3/2} \int dx \frac{x^{1/2}}{e^x - 1}$

$\Rightarrow \quad n_{mag}(T) = \frac{N}{V} = \frac{1}{4\pi^2} \left(\frac{2mT}{\hbar^2}\right)^{3/2} I\left(\frac{1}{2}\right).$

b) $\frac{M(0) - M(T)}{M(0)} = \frac{n_{mag} V\mu}{n V \mu} = \frac{1}{4\pi^2 n} \left(\frac{2mT}{\hbar^2}\right)^{3/2} I\left(\frac{1}{2}\right)$

$= (T/\theta)^{3/2}$

where $\theta^{-3/2} = \frac{1}{4\pi^2 n} \left(\frac{2m}{\hbar^2}\right)^{3/2} I\left(\frac{1}{2}\right)$

$\Rightarrow \quad \theta = \left[\frac{4\pi^2}{I(\frac{1}{2})}\right]^{2/3} \frac{\hbar^2}{2m} n^{2/3}.$

(same as Einstein temperature.)

c) In 2D: $N = (\text{Area}) \int \frac{d^2k}{(2\pi)^2} \frac{1}{\exp\left(\frac{\hbar^2 k^2}{2mT}\right) - 1} = A \int \frac{k\,dk}{2\pi} \frac{1}{e^x - 1}$

$x = \frac{\hbar^2 k^2}{2mT} \Rightarrow dx = 2 \frac{\hbar^2}{2mT} k\,dk \Rightarrow N = \frac{A}{4\pi} \left(\frac{2mT}{\hbar^2}\right) \int dx \frac{1}{e^x - 1}.$

For x small $\frac{N}{A} \propto \int_0 \frac{dx}{x} \Rightarrow$ "IR catastrophe".

3a) $g_R = \exp[\sigma(U_0 - \varepsilon)]$, and we expand

$$\sigma(U_0 - \varepsilon) = \sigma(U_0) - \varepsilon \left.\frac{d\sigma}{dU}\right|_{U=U_0} + \frac{1}{2}\varepsilon^2 \left.\frac{d^2\sigma}{\partial U^2}\right|_{U=U_0} + \cdots$$

$\frac{d\sigma}{dU} = \frac{1}{\tau}$ where τ is temperature of reservoir.

$$\frac{d^2\sigma}{dU^2} = -\frac{1}{\tau^2}\frac{d\tau}{dU} = -\frac{1}{C\tau^2} \quad \text{where } C = \frac{dU}{d\tau}$$

Thus $g_R = e^{\sigma(U_0)} \times \exp\left(-\varepsilon/\tau - \frac{1}{2C}(\varepsilon/\tau)^2 + \cdots\right)$

This factor is a constant, independent of ε

This is the "better Boltzmann factor"

The probability system is in state with energy ε is:

$$P(\varepsilon) = \tilde{Z}^{-1}\exp\left(-\varepsilon/\tau - \frac{1}{2C}(\varepsilon/\tau)^2\right),$$

where $\tilde{Z}$ is a constant, independent of ε.

b) $$\tilde{Z}_1 = L\int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{-|\hbar k c|/\tau}\, e^{-\frac{1}{2C}\left(\frac{\hbar k c}{\tau}\right)^2}$$

$$= \frac{L}{\pi}\frac{\tau}{\hbar c}\int_0^\infty d\left(\frac{\hbar k c}{\tau}\right) e^{-\hbar k c/\tau}\left(1 - \frac{1}{2C}\left(\frac{\hbar k c}{\tau}\right)^2 + \cdots\right)$$

$$= \frac{L}{\pi}\frac{\tau}{\hbar c}\int_0^\infty dx\, e^{-x}\left(1 - \frac{1}{2C}x^2 + \cdots\right)$$

$$= \frac{L}{\pi}\left(\frac{\tau}{\hbar c}\right)\left(1 - \frac{1}{C}\right) = Z_1 + \frac{1}{C}\tilde{Z}_1' \quad \text{where}$$

$$Z_1 = \frac{L}{\pi}\left(\frac{\tau}{\hbar c}\right) = -\tilde{Z}_1'$$

c) Continue to include only the first $O(\frac{1}{C})$ correction:

$$\tilde{Z}_N = \frac{1}{N!}\left(\frac{L}{\pi}\right)^N \int_0^\infty dk_1 \cdots \int_0^\infty dk_N \exp\left[\frac{-\hbar c}{\tau}(k_1+k_2+\cdots+k_N)\right]$$
$$\times\left[1-\frac{1}{2C}\left(\frac{\hbar c}{\tau}\right)^2(k_1+k_2+\cdots+k_N)^2\right]$$

$$= \frac{1}{N!}\left(\frac{L\tau}{\pi\hbar c}\right)^N \int_0^\infty dx_1 \cdots \int_0^\infty dx_N\, e^{-(x_1+x_2+\cdots x_N)}$$
$$\times\left[1-\frac{1}{2C}(x_1+x_2+\cdots x_N)^2\right]$$

$$= \frac{1}{N!}\left(\frac{L\tau}{\pi\hbar c}\right)^N \int_0^\infty dx_1\cdots dx_N\, e^{-x_1}\cdots e^{-x_N}$$
$$\times\left[1-\frac{1}{2C}\left(\sum_{i=1}^N x_i^2+\sum_{i\neq j}x_i x_j\right)\right]$$

Use: $\int_0^\infty dx\, e^{-x}x^2 = 2$

$\int_0^\infty dx_i\, dx_j\, e^{-x_i}e^{-x_j}x_i x_j = 1$

$$= \frac{1}{N!}\left(\frac{L\tau}{\pi\hbar c}\right)^N\left[1-\frac{1}{2C}\left(2N+N(N-1)\right)\right]$$

$$\Longrightarrow \tilde{Z}_N = \frac{1}{N!}\left(\frac{L\tau}{\pi\hbar c}\right)^N\left(1-\frac{1}{2C}N(N+1)\right).$$

d) $$-\frac{\tilde{F}_N}{\tau} = \ln\tilde{Z}_N \cong -N\ln N + N + N\ln\left(\frac{L\tau}{\pi\hbar c}\right) - \frac{1}{2C}N(N+1)$$

$$= N\ln\left(\frac{L\tau}{N\pi\hbar c}\right) + N - \frac{1}{2C}N(N+1)$$

$$= N\ln\left(\frac{n_Q}{n}\right) + N - \frac{1}{2C}N(N+1)$$

where $n = N/L$ and $n_Q = \tau/\pi\hbar c$

$$\Longrightarrow \quad \tilde{F}_N = N\tau\left[\ln\left(\frac{n}{n_Q}\right) - 1 + \frac{N+1}{2C}\right].$$

$$\tilde{\sigma}_N = -\left(\frac{\partial}{\partial \tau}\tilde{F}_N\right)_{C,N} = -\frac{\tilde{F}_N}{\tau} + N\tau\frac{\partial}{\partial \tau}\ln\left(\frac{n_Q}{n}\right)$$

$$= N\left(\ln\left(\frac{n_Q}{n}\right) + 1 - \frac{N+1}{2C}\right) + N$$

$$= N\left[\ln\left(\frac{n_Q}{n}\right) + 2\right] - \frac{N(N+1)}{2C}.$$

The correction is "superextensive" (scales like square of particle number) but still is small if the reservoir is sufficiently large.

Because parts (c) and (d) of this problem were difficult, they were graded leniently.