Ph 12C Final Exam
Due: Friday, 10 June 2016, 5pm

- This exam is to be taken in one continuous time interval not to exceed **4 hours**, beginning when you first open the exam. (You may take one 15 minute break during the exam, which does not count as part of the 4 hours.)

- You may consult the textbook *Thermal Physics* by Kittel and Kroemer, your lecture notes, the online lecture notes, and the problem sets and solutions. If you wish, you may use a calculator or computer for doing calculations. **No other materials or persons are to be consulted.**

- There are four problems, each with multiple parts, and 100 possible points; the value of each problem is indicated. You are to work all of the problems.

- The completed exam is to be handed in at the Ph 12 in-box in East Bridge. All exams are due at 5pm on Friday June 10. **No late exams will be accepted.**

- Good luck!

- Please respond to the TQFR survey. We value your feedback!

1. First quantum correction to pressure – 30 total points

The pressure of an ideal gas can be expanded as a power series in n/n_Q, where $n = N/V$ is the concentration and $n_Q = (m\tau/2\pi\hbar^2)^{3/2}$ is the quantum concentration. The first two terms in this expansion have the form

$$p = n\tau \left(1 + \alpha \frac{n}{n_Q} + \cdots\right). \tag{1}$$

In this problem you will compute the number α, for both fermions and bosons.

(a) (5 points) The (Fermi-Dirac or Bose-Einstein) distribution function $f(\epsilon)$ can be expanded as a power series in $\exp(-(\epsilon - \mu)/\tau)$. Write down the first two terms in this expansion, for both fermions and bosons.

(b) (10 points) Using the approximation to $f(\epsilon)$ from (a), evaluate the particle number

$$N = \int d\epsilon \ \mathcal{D}(\epsilon) f(\epsilon),$$

where $\mathcal{D}(\epsilon)$ is the density of orbitals. Now calculate the chemical potential μ, including the first quantum correction, for both fermions and bosons. Check that the leading term is what you expected.

(c) (10 points) Recalling that $\mu = (\partial F/\partial N)_{\tau,V}$, integrate to find $F(\tau, V, N)$, including the first quantum correction, for both fermions and bosons.

(d) (5 points) Now calculate the pressure, obtaining eq.(1), and find α, for both fermions and bosons. In one case the first quantum correction makes the pressure smaller than for a classical ideal gas with the same concentration and temperature, and in the other case the first quantum correction makes the pressure larger. Are the results what you expected?

2. Fermions in a harmonic potential well — 20 total points

$N \ggg 1$ noninteracting fermions are in a one-dimensional harmonic potential well with circular frequency ω. Assume that all the fermions have spin up, so we don't have to worry about the spin when counting states. Each orbital (harmonic oscillator energy eigenstate) is either empty or singly occupied.

The N lowest orbitals are occupied in the ground state; denote the energies of these occupied orbitals by $E_N^0 > E_{N-1}^0 > E_{N-2}^0 > \cdots > E_1^0$. In an excited state, the occupied orbitals have energies $E_N > E_{N-1} > E_{N-2} > \cdots > E_1$. Hence, relative to the ground state, the excited state has total energy

$$E = (E_N - E_N^0) + (E_{N-1} - E_{N-1}^0) + \cdots + (E_1 - E_1^0).$$

A *partition* of a positive integer n is a way of writing n as a sum of positive integers, and the function $p(n)$ is the number of distinct partitions of n. For example, 4 has five distinct partitions: $4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. Hence $p(4) = 5$

Counting partitions is a notoriously hard combinatorial problem, but as depicted in the recent film *The Man Who Knew Infinity* starring Dev Patel and Jeremy Irons, Ramanujan and Hardy, after a long struggle, found an formula for $p(n)$ which applies asymptotically for very large n:

$$p(n) \approx \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

You may use this formula to compute the thermodynamic properties at high temperature of fermions in a harmonic well.

(*a*) (5 points) Denote by $g(n)$ the number of excited states with energy $E = n\hbar\omega$, and find how $g(n)$ is related to $p(n)$. Assume $N \geq n$.

(*b*) (10 points) Assuming $N \geq n \ggg 1$, find the entropy $\sigma(n)$ and temperature $\tau(n)$, where $E = n\hbar\omega$.

(*c*) (5 points) Express the heat capacity $C = dE/d\tau$ in terms of the temperature τ.

3. Efficiency of a heat engine at peak power — 30 total points

To achieve the optimal Carnot efficiency, a heat engine needs to operate very slowly. A slow engine wastes less heat than a fast one, but on the other hand has lower power output. In this problem we will analyze the efficiency of a heat engine that produces optimal power.

As usual, we assume that the engine operates between two reservoirs with temperatures τ_h and τ_l, where $\tau_h > \tau_l$. The working fluid undergoes a Carnot cycle, but where the temperatures during the two isothermal strokes are τ_{hw} and τ_{lw} such that

$$\tau_h > \tau_{hw} > \tau_{lw} > \tau_l.$$

During the hot isothermal stroke, the *rate* of heat flow from hot reservoir to working fluid is proportional to the temperature difference $x = \tau_h - \tau_{hw}$. Likewise, during the cold isothermal stroke the rate of heat flow from the working fluid to the cold reservoir is proportional to the temperature difference $y = \tau_{lw} - \tau_l$. Thus, increasing these temperature differences increases the rate of heat flow (and hence the power), but also increases the amount of *irreversible* heat flow, compromising efficiency.

Let's assume that both isothermal heat strokes take time t, and that both adiabatic strokes take negligible time. Therefore the heat flow during the hot isothermal stroke is $Q_h = Ktx$, and the heat flow during the cold isothermal stroke is $Q_l = Kty$, for some constant K. (We could relax these assumptions, allowing the hot and cold strokes to take different amounts of time and have different values of K, and allowing the adiabatic strokes to take nonnegligible time. It would turn out, though, that our final answer for the efficiency at peak power would be the same; these simplifying assumptions make the algebra easier.)

(a) (5 points) Express the engines's power output in terms of K, x, and y.

(b) (5 points) Assuming that the working fluid undergoes an ideal Carnot cycle between temperatures τ_{hw} and τ_{lw}, express y, and the power output from (a), in terms of x, τ_h, and τ_l.

(c) (15 points) For fixed τ_h and τ_l, find the value of x that maximizes the power. For this optimal value of x, find also y, τ_{hw} and τ_{lw}. **Hint**: You'll have to solve a quadratic equation, which has two solutions. Be careful to pick the physically relevant solution.

(d) (5 points) What is the efficiency η of the heat engine when it operates at peak power? Compare to the Carnot efficiency η_C.

4. Fun with partial derivatives — 20 total points

Recall that heat capacities at constant volume and pressure are defined as

$$C_V = \tau \left(\frac{\partial \sigma}{\partial \tau}\right)_V, \quad C_P = \tau \left(\frac{\partial \sigma}{\partial \tau}\right)_p.$$

In this problem you will relate the difference $C_P - C_V$ to two other experimentally measurable quantities, the isothermal compressibility

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_\tau$$

and the isobaric thermal expansion coefficient

$$\beta_P = \frac{1}{V}\left(\frac{\partial V}{\partial \tau}\right)_p.$$

(*a*) (5 points) If we regard the entropy σ as a function of τ and V, then

$$d\sigma = \left(\frac{\partial \sigma}{\partial \tau}\right)_V d\tau + \left(\frac{\partial \sigma}{\partial V}\right)_\tau dV, \tag{2}$$

and if we regard the volume V as a function of τ and p, then

$$dV = \left(\frac{\partial V}{\partial \tau}\right)_p d\tau + \left(\frac{\partial V}{\partial p}\right)_\tau dp, \tag{3}$$

Use eq.(2) and eq.(3) to find an expression for $(\partial\sigma/\partial\tau)_p$.

(b) (5 points) Using the result from (*a*) and a Maxwell relation, express $C_P - C_V$ in terms of τ, $(\partial V/\partial \tau)_p$, and $(\partial p/\partial \tau)_V$. **Hint**: To find the relevant Maxwell relation, think about the partial derivatives of $F(\tau, V)$.

(c) (5 points) Using eq.(3), express $(\partial p/\partial \tau)_V$ in terms of $(\partial V/\partial \tau)_p$ and $(\partial V/\partial p)_\tau$

(d) (5 points) Express $C_P - C_V$ in terms of τ, V, K_T and β_P.