Ph 12c -- Final Exam Solution -- 2016

Note Title 6/2/2016

1) First quantum correction to pressure

a) $f(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\mu)/\tau} \pm 1} = \dfrac{1}{e^{(\varepsilon-\mu)/\tau}\left(1 \pm e^{-(\varepsilon-\mu)/\tau}\right)}$

$\simeq e^{-(\varepsilon-\mu)/\tau}\left(1 \mp e^{-(\varepsilon-\mu)/\tau}\right) = e^{-(\varepsilon-\mu)/\tau} \mp e^{-2(\varepsilon-\mu)/\tau}$ $\quad$ − Fermi, + Bose

b) $N = \int_0^\infty d\varepsilon\, D(\varepsilon) = \dfrac{V}{4\pi^2}\left(\dfrac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\varepsilon\, \varepsilon^{\frac{1}{2}}\left(e^{-(\varepsilon-\mu)/\tau} \mp e^{-2(\varepsilon-\mu)/\tau}\right).$

$\int_0^\infty d\varepsilon\, \varepsilon^{\frac{1}{2}} e^{-\varepsilon/\tau} = \tau^{3/2}\int_0^\infty dy\, y^{\frac{1}{2}} e^{-y}$ $\quad$ Let $y = x^2$, $dy = 2x\,dx$

$= \tau^{3/2}\int_0^\infty 2x^2 e^{-x^2} = \tau^{3/2}\dfrac{\sqrt{\pi}}{2}$

$\int_0^\infty d\varepsilon\, \varepsilon^{\frac{1}{2}} e^{-2\varepsilon/\tau} = \left(\dfrac{\tau}{2}\right)^{3/2}\int_0^\infty dy\, y^{\frac{1}{2}} e^{-y} = \left(\dfrac{\tau}{2}\right)^{3/2}\dfrac{\sqrt{\pi}}{2}$

Therefore, $n = \dfrac{N}{V} = \dfrac{1}{4\pi^2}\left(\dfrac{2m}{\hbar^2}\right)^{3/2}\dfrac{\sqrt{\pi}}{2}\tau^{3/2}\left(e^{\mu/\tau} \mp \dfrac{1}{2^{3/2}} e^{2\mu/\tau}\right) + \cdots$

$= n_Q e^{\mu/\tau}\left(1 \mp \dfrac{1}{2^{3/2}} e^{\mu/\tau}\right) + \cdots$

Thus $e^{\mu/\tau} = \dfrac{n}{n_Q} + \text{h.o.} \Rightarrow e^{\mu/\tau} = \dfrac{n}{n_Q}\left(1 \pm \dfrac{1}{2^{3/2}}\dfrac{n}{n_Q}\right) + \cdots$

$\Rightarrow \mu/\tau = \ln\dfrac{n}{n_Q} + \ln\left(1 \pm \dfrac{1}{2^{3/2}}\dfrac{n}{n_Q}\right)$

$= \ln\dfrac{n}{n_Q} \pm \dfrac{1}{2^{3/2}}\dfrac{n}{n_Q}$

$\Rightarrow \mu = \tau\left(\ln\dfrac{n}{n_Q} \pm \dfrac{1}{2^{3/2}}\dfrac{n}{n_Q}\right)$ $\quad$ + Fermi, − Bose

c) $F = \int \mu\, dN$. Since $\int_0^N dN \ln N = N\ln N - N$

$\int_0^N dN\, N = \frac{1}{2}N^2,$

$F = \tau\left(\ln\dfrac{1}{Vn_Q} + N\ln N - N \pm \dfrac{1}{2^{5/2}}\dfrac{N^2}{Vn_Q}\right)$

$$\Rightarrow \quad F = NT\left(\ln\frac{n}{n_Q} - 1 \pm \frac{1}{2^{5/2}}\frac{n}{n_Q}\right)$$

d) $$p = -\frac{\partial F}{\partial V} = NT\left(\frac{1}{V} \pm \frac{1}{2^{5/2}}\frac{n}{Vn_Q}\right)$$

$$\Rightarrow \quad p = \frac{NT}{V}\left(1 \pm \frac{1}{4\sqrt{2}}\frac{n}{n_Q}\right)$$

+ Fermi
− Bose

The first quantum correction makes the pressure larger for fermions and smaller for bosons. That makes sense; fermions want to avoid one another, while bosons want to be together.

2) Fermions in a harmonic potential well

a) Define the zero of energy so that $E_j^0 = j\hbar\omega$.

Suppose in the excited state $E_j = n_j\hbar\omega$ where $n_j \geq j$

Then $$E/\hbar\omega = n = \sum_{j=1}^{N}(n_j - j).$$

Here we have expressed n as a sum of nonnegative integers listed in nonascending order — each integer is no larger than the preceding one. That is because $n_j \geq n_{j-1}+1$ and therefore $(n_j - j) \geq (n_{j-1} - j-1)$.

The positive integers in the sum provide a partition of n; furthermore, each partition of n arises in some possible state (assuming $N \geq n$) and no two partitions correspond to the same state. Therefore,

$$g(n) = p(n)$$

b) $$g(n) = p(n) \approx \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}\,n}$$

$$\Rightarrow \quad \sigma(n) = \ln g(n) = \pi\sqrt{2n/3} + \cdots$$

and $$\frac{1}{\tau} = \frac{\partial\sigma}{\partial E} = \frac{\partial\sigma}{\partial(\hbar\omega n)} = \frac{1}{\hbar\omega}\,\pi\sqrt{\frac{2}{3}}\,\frac{1}{2}n^{-\frac{1}{2}}$$

$$\Rightarrow \quad \tau = \hbar\omega\frac{\sqrt{6n}}{\pi}$$

c) $$\frac{1}{C} = \frac{\partial\tau}{\partial E} = \frac{\partial}{\partial n}\left(\frac{\sqrt{6n}}{\pi}\right) = \frac{1}{\pi}\sqrt{\frac{3}{2}}\,n^{-1/2}$$

$$\Rightarrow \quad C = \pi\sqrt{\frac{2n}{3}} \Rightarrow C = \frac{\pi^2}{3}\tau/\hbar\omega$$

3) Efficiency of a heat engine at peak power

a) $Q_h = Ktx$, $Q_l = Kty$ $\Rightarrow$

Work $W = Q_h - Q_l = Kt(x-y)$ where $x = \tau_h - \tau_{hw}$, $y = \tau_{lw} - \tau_l$

and cycle takes time $2t$ $\Rightarrow$

$$\text{Power} = \frac{\text{work}}{\text{time}} = \frac{Kt(x-y)}{2t} = \frac{1}{2}K(x-y)$$

b) Carnot cycle for working fluid $\Rightarrow$

$$\frac{Q_h}{\tau_{hw}} = \frac{Q_l}{\tau_{lw}} \Rightarrow \frac{KtX}{\tau_h - X} = \frac{Kty}{\tau_l + y}$$

$$\Rightarrow (\tau_l + y)X = (\tau_h - X)y \Rightarrow \tau_l X = (\tau_h - 2X)y$$

$$\Rightarrow \quad y = \frac{\tau_l X}{\tau_h - 2X}$$

c) We are to maximize power $\propto$ $x - y = X - \dfrac{\tau_l X}{\tau_h - 2X}$

$$\Rightarrow \frac{d}{dx}(x-y) = 1 - \frac{\tau_l}{\tau_h - 2x} - \frac{2\tau_l X}{(\tau_h - 2x)^2} = 0$$

$$\Rightarrow \quad (T_h - 2x)^2 - T_\ell (T_h - 2x) - 2T_\ell x = 0$$
$$= 4x^2 + (-4T_h + 2T_\ell - 2T_\ell)x + T_h^2 - T_h T_\ell$$
$$\Rightarrow \quad 0 = x^2 - T_h x + \tfrac{1}{4}(T_h^2 - T_h T_\ell)$$
$$\Rightarrow \quad x = \tfrac{1}{2}\left(T_h \pm \sqrt{T_h^2 - T_h^2 + T_h T_\ell}\right)$$
$$= \tfrac{1}{2}\left(T_h \pm \sqrt{T_h T_\ell}\right)$$

Therefore $y = \dfrac{T_\ell x}{T_h - (T_h \pm \sqrt{T_h T_\ell})} = \mp \sqrt{\dfrac{T_\ell}{T_h}}\, x$.

we want the solution with $y > 0 \Rightarrow$

$$x = \tfrac{1}{2}\left(T_h - \sqrt{T_h T_\ell}\right), \quad y = \tfrac{1}{2}\left(\sqrt{T_h T_\ell} - T_\ell\right)$$

$$T_{hw} = T_h - x = \tfrac{1}{2}\left(T_h + \sqrt{T_h T_\ell}\right)$$
$$T_{\ell w} = T_\ell + y = \tfrac{1}{2}\left(T_\ell + \sqrt{T_h T_\ell}\right)$$

d) Efficiency is $\eta = \dfrac{W}{Q_h} = \dfrac{Kt(x-y)}{Ktx} = 1 - \dfrac{y}{x}$

$$\Rightarrow \quad \eta = 1 - \sqrt{\frac{T_\ell}{T_h}}.$$

Compare with $\eta_c = 1 - \dfrac{T_\ell}{T_h}$:

$$\Rightarrow \quad \frac{\eta_c}{\eta} = 1 + \sqrt{T_\ell / T_h},$$

which is > 1 for $T_\ell > 0$, and approaches 2 as $T_\ell \to T_h$.

Real heat engines have efficiency reasonably close to η.

4) Fun with partial derivatives

a) $$d\sigma = \left(\frac{\partial \sigma}{\partial T}\right)_V dT + \left(\frac{\partial \sigma}{\partial V}\right)_T dV$$

and $$dV = \left(\frac{\partial V}{\partial T}\right)_p dT + \left(\frac{\partial V}{\partial p}\right)_T dp$$

$$\Rightarrow d\sigma = \left(\frac{d \sigma}{d T}\right)_V dT + \left(\frac{\partial \sigma}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p dT + \left(\frac{\partial \sigma}{\partial V}\right)_T \left(\frac{\partial V}{\partial p}\right)_T dp.$$

Holding p fixed:

$$\left(\frac{\partial \sigma}{\partial T}\right)_p = \left(\frac{\partial \sigma}{\partial T}\right)_V + \left(\frac{\partial \sigma}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p$$

b) To eliminate $\left(\frac{\partial \sigma}{\partial V}\right)_T$ in favor of $\left(\frac{\partial p}{\partial T}\right)_V$,

consider $$dF = -\sigma\, dT - p\, dV$$

Because $\left(\frac{\partial}{\partial T}\right)_V \left(\frac{\partial}{\partial V}\right)_T$ commute, we have

$$\left(\frac{\partial \sigma}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V \Rightarrow$$

$$C_p - C_V = T\left(\left(\frac{\partial \sigma}{\partial T}\right)_p - \left(\frac{\partial \sigma}{\partial T}\right)_V\right) = T\left(\frac{\partial \sigma}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p$$

$$\Rightarrow C_p - C_V = T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p$$

c) $$dV = \left(\frac{\partial V}{\partial T}\right)_p dT + \left(\frac{\partial V}{\partial p}\right)_T dp = 0 \quad \text{(when } V \text{ fixed).}$$

$$\Rightarrow \left(\frac{dp}{dT}\right)_V = -\frac{(\partial V/\partial T)_p}{(\partial V/\partial p)_T}$$

d) Plug (c) into (b):

$$C_p - C_V = -T\,\frac{(\partial V/\partial T)_p^2}{(\partial V/\partial p)_T}.$$

Since $K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$, $\beta_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$,

$$C_P - C_V = TV\frac{\beta_P^2}{K_T}.$$