

Ph 12c Midterm Exam
Due: Thursday, 5 May 2016, 8pm

- This exam is to be taken in one continuous time interval not to exceed **3 hours**, beginning when you first open the exam. (You may take one 15 minute break during the exam, which does not count as part of the 3 hours.)
- You may consult the textbook *Thermal Physics* by Kittel and Kroemer, your lecture notes, the online lecture notes, and the problem sets and solutions. If you wish, you may use a calculator or computer for doing calculations. However, this probably won't be necessary. **No other materials or persons are to be consulted.**
- There are three problems, each with multiple parts, and 100 possible points; the value of each problem is indicated. You are to work all of the problems.
- The completed exam is to be handed in at the Ph 12 in-box in East Bridge. All exams are due at 8pm on Thursday, May 5. **No late exams will be accepted.**
- Good luck!

1. Increase of entropy — 30 total points

Consider two identical systems \mathcal{S}_1 and \mathcal{S}_2 . For both systems, the internal energy U is related to the temperature τ by $U = C\tau$, where C is a temperature-independent constant (the same constant for both systems). Initially each system is isolated — the temperature of \mathcal{S}_1 is τ_1 and the temperature of \mathcal{S}_2 is τ_2 , where $\tau_1 > \tau_2$. Then the systems are brought into thermal contact, and eventually come to thermal equilibrium at the final temperature τ .

- (a) (5 points) Find the final equilibrium temperature τ .
- (b) (15 points) Find the change $\Delta\sigma$ in the *total* entropy due to the flow of heat from \mathcal{S}_1 to \mathcal{S}_2 .
- (c) (10 points) Show that $\Delta\sigma > 0$.

2. The number of magnons — 35 total points

A ferromagnet has excitations called *magnons*, which are quantized spin waves. Conceptually, magnons are analogous to phonons, but with two important differences. (1) While a phonon with wavenumber \vec{k} has three polarizations, a magnon has only one. (2) While the energy of a phonon with wave number \vec{k} is $\epsilon = \hbar|\vec{k}|v$, where v is the speed of sound, magnons obey a nonrelativistic dispersion relation $\epsilon = (\hbar k)^2/2m$, where m is the magnon's *effective mass*. As with phonons, there is a short-distance cutoff on the wavelength of magnons, but you may ignore the cutoff for this problem, because we will be considering magnons at sufficiently low temperature that the cutoff is not important.

- (a) (15 points) For a three-dimensional ferromagnet at low temperature, calculate the total number $n_{\text{mag}}(\tau)$ of magnons per unit volume. The answer involves a dimensionless integral, which you need not evaluate numerically. Instead, express your answer in terms of

$$I(\alpha) = \int_0^\infty dx \frac{x^\alpha}{e^x - 1}. \quad (1)$$

for some value of α .

At zero temperature, all the spins in the ferromagnet are aligned, so the total magnetization is $M(\tau = 0) = nV\mu$, where μ is the magnetic moment of a single spin, n is the concentration of spins, and V is the volume of the ferromagnet. At nonzero temperature, each magnon reduces the magnetization by μ , so that $M(\tau) = M(0) - n_{\text{mag}}(\tau)V\mu$.

- (b) (10 points) Show that

$$\frac{M(0) - M(\tau)}{M(0)} = \left(\frac{\tau}{\theta}\right)^\beta; \quad (2)$$

Find the power β and the temperature θ .

According to the formula eq.(2), the magnetization $M(\tau)$ hits zero at $\tau = \theta$. We interpret this to mean that for $\tau \geq \theta$ the ferromagnet demagnetizes. For temperatures above θ there are no spin waves, and our formula for $M(\tau)$ can no longer be trusted.

- (c) (10 points) Show that in a two-dimensional ferromagnet the number of magnons per unit area *diverges* at any nonzero temperature τ . This “infrared catastrophe” indicates that a two-dimensional ferromagnet has no spontaneous magnetization for $\tau > 0$.

3. Better Boltzmann factor — 35 total points

Consider a system \mathcal{S} that is in thermal equilibrium with a large reservoir \mathcal{R} . Then, according to the fundamental assumption of thermal physics, the probability $P(\epsilon)$ that the system is in a specified state with energy ϵ is proportional to $g_{\mathcal{R}}(U_0 - \epsilon)$; here, $g_{\mathcal{R}}(U)$ is the number of accessible states of the reservoir with energy U , and U_0 is the total energy shared by the system and the reservoir.

We derived the Boltzmann factor by expanding $\sigma_{\mathcal{R}} = \ln[g_{\mathcal{R}}(U_0 - \epsilon)]$ to linear order in ϵ . Corrections to the Boltzmann factor can be obtained by carrying out the expansion to higher order in ϵ .

- (a) (10 points) Find the corrected Boltzmann factor, retaining terms up to quadratic order in ϵ . Express the answer in terms of ϵ , the temperature of the reservoir τ , and the heat capacity of the reservoir $C = (\partial U / \partial \tau)_{V,N}$.

The heat capacity C is an extensive quantity, whose value is proportional to the size of the reservoir. Usually we may consider C to be so large that the correction to the Boltzmann factor found in (a) can be safely neglected. But for a sufficiently small reservoir, these corrections might be significant.

- (b) (15 points) Suppose now that the system \mathcal{S} is a single relativistic particle, with energy $\epsilon = \hbar k c$, in a one-dimensional box with length L . Using the improved Boltzmann factor from (a), compute the partition function \tilde{Z}_1 for this one-particle “gas.” Don’t calculate \tilde{Z}_1 exactly; just find the leading correction in a power series expansion in $1/C$. That is, write

$$\tilde{Z}_1 = Z_1 + \frac{1}{C} \tilde{Z}'_1 + \text{higher order}, \quad (3)$$

and find Z_1 and \tilde{Z}'_1 . Here Z_1 is the partition function in the limit $C \rightarrow \infty$, the next term is the correction linear in $1/C$, and “higher order” means terms suppressed by additional powers of $1/C$ (which you need not calculate). Assume the box is sufficiently large so that the sum over states can be approximated by an integral over the wave number k . **Hint:** You might find it useful to recall that

$$\int_0^\infty x^p e^{-x} = p! \quad (4)$$

for any nonnegative integer p .

- (c) (5 points) Now use \tilde{Z}_1 to calculate \tilde{Z}_N , the partition function for N indistinguishable relativistic particles in the box of length L . Assume the gas is in the classical regime, so you don’t have to worry about any of the orbitals in the box being occupied by more than one particle.
- (d) (5 points) Using the expression for \tilde{Z}_N from (c), find the free energy \tilde{F}_N and entropy $\tilde{\sigma}_N$ of this one-dimensional relativistic gas.