Ph 219b/CS 219b

Exercises
Due: Wednesday 20 November 2013

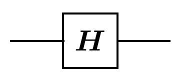
3.1 Universal quantum gates I

In this exercise and the two that follow, we will establish that several simple sets of gates are universal for quantum computation.

The *Hadamard transformation* $\boldsymbol{H}$ is the single-qubit gate that acts in the standard basis $\{|0\rangle, |1\rangle\}$ as

$$\boldsymbol{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; ; \tag{1}$$

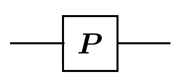
in quantum circuit notation, we denote the Hadamard gate as



The single-qubit *phase gate* $\boldsymbol{P}$ acts in the standard basis as

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \; , \tag{2}$$

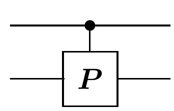
and is denoted



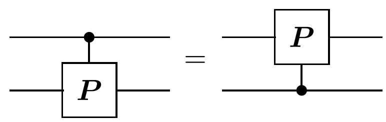
A two-qubit *controlled phase gate* $\Lambda(\boldsymbol{P})$ acts in the standard basis $\{|00\rangle, 01\rangle, |10\rangle, |11\rangle\}$ as the diagonal 4×4 matrix

$$\Lambda(\boldsymbol{P}) = \mathrm{diag}(1, 1, 1, i) \tag{3}$$

and can be denoted

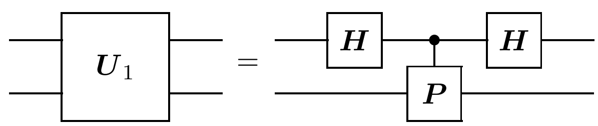


Despite this misleading notation, the gate $\Lambda(\boldsymbol{P})$ actually acts symmetrically on the two qubits:

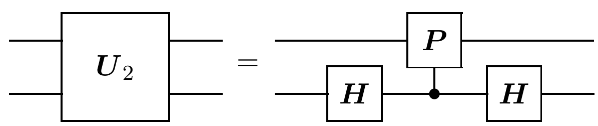


We will see that the two gates $\boldsymbol{H}$ and $\Lambda(\boldsymbol{P})$ comprise a *universal gate set* – any unitary transformation can be approximated to arbitrary accuracy by a quantum circuit built out of these gates.

a) Consider the two-qubit unitary transformations $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ defined by quantum circuits



and



Let $|ab\rangle$ denote the element of the standard basis where a labels the upper qubit in the circuit diagram and b labels the lower qubit. Write out $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ as 4×4 matrices in the standard basis. Show that $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ both act trivially on the states

$$|00\rangle, \quad \frac{1}{\sqrt{3}}\left(|01\rangle + |10\rangle + |11\rangle\right) \ . \tag{4}$$

b) Thus $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ act nontrivially only in the two-dimensional space spanned by

$$\left\{\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right), \frac{1}{\sqrt{6}}\left(|01\rangle + |10\rangle - 2|11\rangle\right)\right\} \ . \tag{5}$$

Show that, expressed in this basis, they are

$$\boldsymbol{U}_1 = \frac{1}{4}\begin{pmatrix} 3+i & \sqrt{3}(-1+i) \\ \sqrt{3}(-1+i) & 1+3i \end{pmatrix} \ , \tag{6}$$

and

$$\boldsymbol{U}_2 = \frac{1}{4}\begin{pmatrix} 3+i & \sqrt{3}(1-i) \\ \sqrt{3}(1-i) & 1+3i \end{pmatrix} \ . \tag{7}$$

c) Now express the action of $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ on this two-dimensional subspace in the form

$$\boldsymbol{U}_1 = \sqrt{i}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\hat{n}_1 \cdot \vec{\boldsymbol{\sigma}}\right) \ , \tag{8}$$

and

$$\boldsymbol{U}_2 = \sqrt{i}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\hat{n}_2 \cdot \vec{\boldsymbol{\sigma}}\right) \ . \tag{9}$$

What are the unit vectors $\hat{n}_1$ and $\hat{n}_2$?

d) Consider the transformation $\boldsymbol{U}_2^{-1}\boldsymbol{U}_1$ (Note that $\boldsymbol{U}_2^{-1}$ can also be constructed from the gates $\boldsymbol{H}$ and $\Lambda(\boldsymbol{P})$.) Show that it performs a rotation with half-angle $\theta/2$ in the two-dimensional space spanned by the basis eq. (5), where $\cos(\theta/2) = 1/4$.

3.2 Universal quantum gates II

We have now seen how to compose our fundamental quantum gates to perform, in a two-dimensional subspace of the four-dimensional Hilbert space of two qubits, a rotation with $\cos(\theta/2) = 1/4$. In this exercise, we will show that the angle θ is not a rational multiple of π. Equivalently, we will show that

$$e^{i\theta/2} \equiv \cos(\theta/2) + i\sin(\theta/2) = \frac{1}{4}\left(1 + i\sqrt{15}\right) \tag{10}$$

is not a root of unity: there is no finite integer power n such that $(e^{i\theta/2})^n = 1$.

Recall that a *polynomial of degree* n is an expression

$$P(x) = \sum_{k=0}^{n} a_k x^k \tag{11}$$

with $a_n \neq 0$. We say that the polynomial is *rational* if all of the a_k's are rational numbers, and that it is *monic* if $a_n = 1$. A polynomial is *integral* if all of the a_k's are integers, and an integral polynomial is *primitive* if the greatest common divisor of $\{a_0, a_1, \ldots, a_n\}$ is 1.

a) Show that the monic rational polynomial of minimal degree that has $e^{i\theta/2}$ as a root is

$$P(x) = x^2 - \frac{1}{2}x + 1 \ . \tag{12}$$

The property that $e^{i\theta/2}$ is not a root of unity follows from the result (a) and the

Theorem *If a is a root of unity, and $P(x)$ is a monic rational polynomial of minimal degree with $P(a) = 0$, then $P(x)$ is integral.*

Since the minimal monic rational polynomial with root $e^{i\theta/2}$ is not integral, we conclude that $e^{i\theta/2}$ is not a root of unity. In the rest of this exercise, we will prove the theorem.

b) By "long division" we can prove that if $A(x)$ and $B(x)$ are rational polynomials, then there exist rational polynomials $Q(x)$ and $R(x)$ such that

$$A(x) = B(x)Q(x) + R(x) \ , \tag{13}$$

where the "remainder" $R(x)$ has degree less than the degree of $B(x)$. Suppose that $a^n = 1$, and that $P(x)$ is a rational polynomial of minimal degree such that $P(a) = 0$. Show that there is a rational polynomial $Q(x)$ such that

$$x^n - 1 = P(x)Q(x) \ . \tag{14}$$

c) Show that if $A(x)$ and $B(x)$ are both primitive integral polynomials, then so is their product $C(x) = A(x)B(x)$. **Hint**: If $C(x) = \sum_k c_k x^k$ is not primitive, then there is a prime number p that divides all of the c_k's. Write $A(x) = \sum_l a_l x^l$, and $B(x) = \sum_m b_m x^m$, let a_r denote the coefficient of lowest order in $A(x)$ that is not divisible by p (which must exist if $A(x)$ is primitive), and let b_s denote the coefficient of lowest order in $B(x)$ that is not divisible by p. Express the product $a_r b_s$ in terms of c_{r+s} and the other a_l's and b_m's, and reach a contradiction.

d) Suppose that a monic integral polynomial $P(x)$ can be factored into a product of two monic rational polynomials, $P(x) = A(x)B(x)$. Show that $A(x)$ and $B(x)$ are integral. **Hint:** First note that we may write $A(x) = (1/r)\cdot\tilde{A}(x)$, and $B(x) = (1/s)\cdot\tilde{B}(x)$, where r, s are positive integers, and $\tilde{A}(x)$ and $\tilde{B}(x)$ are primitive integral; then use (c) to show that $r = s = 1$.

e) Combining (b) and (d), prove the theorem.

What have we shown? Since $\boldsymbol{U}_2^{-1}\boldsymbol{U}_1$ is a rotation by an irrational multiple of π, the powers of $\boldsymbol{U}_2^{-1}\boldsymbol{U}_1$ are dense in a $U(1)$ subgroup.

Similar reasoning shows that $\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ is a rotation by the same angle about a different axis, and therefore its powers are dense in another $U(1)$ subgroup. Products of elements of these two noncommuting $U(1)$ subgroups are dense in the $SU(2)$ subgroup that contains both $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$.

Furthermore, products of $\Lambda(\boldsymbol{P})\boldsymbol{U}_2^{-1}\boldsymbol{U}_1\Lambda(\boldsymbol{P})^{-1}$ and $\Lambda(\boldsymbol{P})\boldsymbol{U}_1\boldsymbol{U}_2^{-1}\Lambda(\boldsymbol{P})^{-1}$ are dense in another $SU(2)$, acting on the span of

$$\left\{\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \frac{1}{\sqrt{6}}(|01\rangle + |10\rangle - 2i|11\rangle)\right\} . \qquad (15)$$

Together, these two $SU(2)$ subgroups close on the $SU(3)$ subgroup that acts on the three-dimensional space orthogonal to $|00\rangle$. Conjugating this $SU(3)$ by $\boldsymbol{H}\otimes\boldsymbol{H}$ we obtain another $SU(3)$ acting on the three dimensional space orthogonal to $|+,+\rangle$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$. The only subgroup of $SU(4)$ that contains both of these $SU(3)$ subgroups is $SU(4)$ itself.

Therefore, the circuits constructed from the gate set $\{\boldsymbol{H}, \Lambda(\boldsymbol{P})\}$ are dense in $SU(4)$ — we can approximate any two-qubit gate to arbitrary accuracy, which we know suffices for universal quantum computation. Whew!

3.3 Universal quantum gates III

We have shown that the gate set $\{\boldsymbol{H}, \Lambda(\boldsymbol{P})\}$ is universal. Thus any gate set from which both $\boldsymbol{H}$ and $\Lambda(\boldsymbol{P})$ can be constructed is also universal. In particular, we can see that $\{\boldsymbol{H}, \boldsymbol{P}, \Lambda^2(\boldsymbol{X})\}$ is a universal set.

a) It is sometimes convenient to characterize a quantum gate by specifying the action of the gate when it conjugates a Pauli operator. Show that $\boldsymbol{H}$ and $\boldsymbol{P}$ have the properties

$$\boldsymbol{HXH} = \boldsymbol{Z} , \quad \boldsymbol{HYH} = -\boldsymbol{Y} , \quad \boldsymbol{HZH} = \boldsymbol{X} , \qquad (16)$$

and

$$\boldsymbol{PXP}^{-1} = \boldsymbol{Y} , \quad \boldsymbol{PYP}^{-1} = -\boldsymbol{X} , \quad \boldsymbol{PZP}^{-1} = \boldsymbol{Z} . \qquad (17)$$

b) Note that, since $\boldsymbol{P}^{-1} = \boldsymbol{P}^3$, the gate $\boldsymbol{K} = \boldsymbol{HP}^{-1}\boldsymbol{HPH}$ can be constructed using $\boldsymbol{H}$ and $\boldsymbol{P}$. Show that

$$\boldsymbol{KXK} = \boldsymbol{Y} , \quad \boldsymbol{KYK} = \boldsymbol{X} , \quad \boldsymbol{KZK} = -\boldsymbol{Z} . \qquad (18)$$

c) Construct circuits for $\Lambda^2(\boldsymbol{Y})$ and $\Lambda^2(\boldsymbol{Z})$ using the gate set $\{\boldsymbol{H}, \boldsymbol{P}, \Lambda^2(\boldsymbol{X})\}$. Then complete the proof of universality for this gate set by constructing $\Lambda(\boldsymbol{P}) \otimes \boldsymbol{I}$ using $\Lambda^2(\boldsymbol{X})$, $\Lambda^2(\boldsymbol{Y})$, and $\Lambda^2(\boldsymbol{Z})$.

The Toffoli gate $\Lambda^2(\boldsymbol{X})$ is universal for reversible classical computation. What must be added to realize the full power of quantum computing? We have just seen that the single-qubit gates $\boldsymbol{H}$ and $\boldsymbol{P}$, together with the Toffoli gate, are adequate for reaching any unitary transformation. But in fact, just $\boldsymbol{H}$ and $\Lambda^2(\boldsymbol{X})$ suffice to efficiently simulate any quantum computation.

Of course, since $\boldsymbol{H}$ and $\Lambda^2(\boldsymbol{X})$ are both real orthogonal matrices, a circuit composed from these gates is necessarily real — there are complex n-qubit unitaries that cannot be constructed with these tools. But a 2^n-dimensional complex vector space is isomorphic to a 2^{n+1}–dimensional real vector space. A complex vector can be encoded by a real vector according to

$$|\psi\rangle = \sum_x \psi_x |x\rangle \mapsto |\tilde{\psi}\rangle = \sum_x (\mathrm{Re}\ \psi_x)|x,0\rangle + (\mathrm{Im}\ \psi_x)|x,1\rangle \ , \tag{19}$$

and the action of the unitary transformation $\boldsymbol{U}$ can be represented by a real orthogonal matrix $\tilde{U}_R$ defined as

$$\begin{aligned} U_R : \qquad & |x,0\rangle \mapsto (\mathrm{Re}\ U)|x\rangle \otimes |0\rangle + (\mathrm{Im}\ U)|x\rangle \otimes |1\rangle \ , \\ & |x,1\rangle \mapsto -(\mathrm{Im}\ U)|x\rangle \otimes |0\rangle + (\mathrm{Re}\ U)|x\rangle \otimes |1\rangle \ . \end{aligned} \tag{20}$$

To show that the gate set $\{\boldsymbol{H}, \Lambda^2(\boldsymbol{X})\}$ is "universal," it suffices to demonstrate that the real encoding $\Lambda(\boldsymbol{P})_R$ of $\Lambda(\boldsymbol{P})$ can be constructed from $\Lambda^2(\boldsymbol{X})$ and $\boldsymbol{H}$.

d) Verify that $\Lambda(\boldsymbol{P})_R = \Lambda^2(\boldsymbol{XZ})$.

e) Use $\Lambda^2(\boldsymbol{X})$ and $\boldsymbol{H}$ to construct a circuit for $\Lambda^2(\boldsymbol{XZ})$.

Thus, the classical Toffoli gate does not need much help to unleash the power of quantum computing. In fact, *any* nonclassical single-qubit gate (one that does not preserve the computational basis), combined with the Toffoli gate, is sufficient.

3.4 Universality from any entangling two-qubit gate

We say that a two-qubit unitary quantum gate is *local* if it is a tensor product of single-qubit gates, and that the two-qubit gates $\boldsymbol{U}$ and $\boldsymbol{V}$ are *locally equivalent* if one can be transformed to the other by local gates:

$$\boldsymbol{V} = (\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{U}(\boldsymbol{C} \otimes \boldsymbol{D}) \ . \tag{21}$$

It turns out (you are not asked to prove this) that every two-qubit gate is locally equivalent to a gate of the form:

$$\boldsymbol{V}(\theta_x, \theta_y, \theta_z) = \exp\left[i\left(\theta_x \boldsymbol{X} \otimes \boldsymbol{X} + \theta_y \boldsymbol{Y} \otimes \boldsymbol{Y} + \theta_z \boldsymbol{Z} \otimes \boldsymbol{Z}\right)\right] \ , \tag{22}$$

where

$$-\pi/4 < \theta_x \leq \theta_y \leq \theta_z \leq \pi/4 \ . \tag{23}$$

a) Show that $\boldsymbol{V}(\pi/4, \pi/4, \pi/4)$ is (up to an overall phase) the **SWAP** operation that interchanges the two qubits:

$$\textbf{SWAP}\left(|\psi\rangle \otimes |\phi\rangle\right) = |\phi\rangle \otimes |\psi\rangle \ . \tag{24}$$

b) Show that $\boldsymbol{V}(0, 0, \pi/4)$ is locally equivalent to the CNOT gate $\Lambda(\boldsymbol{X})$.

As discussed in the lecture notes, the CNOT gate $\Lambda(\boldsymbol{X})$ together with arbitrary single-qubit gates form a universal gate set. But in fact there is nothing special about the the CNOT gate in this regard. *Any* two-qubit gate $\boldsymbol{U}$, when combined with arbitrary single-qubit gates, suffices for universality *unless* $\boldsymbol{U}$ is either local or locally equivalent to **SWAP**.

To demonstrate that $\boldsymbol{U}$ is universal when assisted by local gates it suffices to construct $\Lambda(\boldsymbol{X})$ using a circuit containing only local gates and $\boldsymbol{U}$ gates.

Lemma *If* $\boldsymbol{U}$ *is locally equivalent to* $\boldsymbol{V}(\theta_x, \theta_y, \theta_z)$*, then* $\Lambda(\boldsymbol{X})$ *can be constructed from a circuit using local gates and* $\boldsymbol{U}$ *gates except in two cases: (1)* $\theta_x = \theta_y = \theta_z = 0$ *(*$\boldsymbol{U}$ *is local), (2)* $\theta_x = \theta_y = \theta_z = \pi/4$ *(*$\boldsymbol{U}$ *is locally equivalent to* **SWAP***)..*

You will prove the Lemma in the rest of this exercise.

c) Show that:

$$\begin{aligned}(\boldsymbol{I}\otimes\boldsymbol{X})\boldsymbol{V}(\theta_x,\theta_y,\theta_z)(\boldsymbol{I}\otimes\boldsymbol{X})\boldsymbol{V}(\theta_x,\theta_y,\theta_z) &= \boldsymbol{V}(2\theta_x,0,0)\ ,\\ (\boldsymbol{I}\otimes\boldsymbol{Y})\boldsymbol{V}(\theta_x,\theta_y,\theta_z)(\boldsymbol{I}\otimes\boldsymbol{Y})\boldsymbol{V}(\theta_x,\theta_y,\theta_z) &= \boldsymbol{V}(0,2\theta_y,0)\ ,\\ (\boldsymbol{I}\otimes\boldsymbol{Z})\boldsymbol{V}(\theta_x,\theta_y,\theta_z)(\boldsymbol{I}\otimes\boldsymbol{Z})\boldsymbol{V}(\theta_x,\theta_y,\theta_z) &= \boldsymbol{V}(0,0,2\theta_z)\ .\end{aligned} \tag{25}$$

d) Show that $\boldsymbol{V}(0,0,\theta)$ is locally equivalent to the controlled rotation $\Lambda[\boldsymbol{R}(\hat{n},4\theta)]$, where $\boldsymbol{R}(\hat{n},4\theta)=\exp[-2i\theta(\hat{n}\cdot\boldsymbol{\sigma})]$, for an arbitrary axis of rotation $\hat{n}$. (Here $\boldsymbol{\sigma}=(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z})$.)

e) Now use the results of (*c*) and (*d*) to prove the Lemma.

3.5 BQP is contained in PP

We have seen that **BQP** (the class of decision problems that can be solved efficiently by a quantum computer) is contained in the classical complexity class **PSPACE** (the decision problems that can be solved using a polynomial-size memory). The purpose of this problem is to establish an inclusion that is presumed to be stronger: **BQP** is contained in **PP**. A decision problem is in **PP** ("probabilistic polynomial time") if it can be solved in polynomial time by a randomized classical computation with probability of success greater than 1/2. (In contrast to the class **BPP**, the success probability is not required to exceed 1/2 by a positive constant independent of the input size.)

When a decision problem is solved by a quantum computer, one particular qubit may be designated as the "answer qubit" — the qubit that is measured to decide the answer. If the quantum circuit builds the unitary transformation $\boldsymbol{U}$, which acts on the input state $|0\rangle$, then the probability distribution governing the answer bit x can be expressed as

$$P(x)=\sum_y |\langle x,y|\boldsymbol{U}|0\rangle|^2\ , \tag{26}$$

where $|y\rangle$ denotes a basis state for the "junk" output qubits that are not measured. If the problem is in **BQP**, then there is a polynomial-size uniform quantum circuit family such that $P(x)\geq 2/3$ when x is the correct value of the output. For convenience, we are assuming here that the input to the Boolean function being evaluated is encoded by choosing the unitary U to depend on the this input; hence we may suppose that the unitary always acts on the fixed quantum state $|0\rangle$.

a) Show that if a problem is in **BQP** then there is a polynomial-size uniform circuit family (where the circuit depends on the input to the problem) such that $|\langle 0|\boldsymbol{U}|0\rangle|^2 \geq 2/3$ if the correct output is 0 and $|\langle 0|\boldsymbol{U}|0\rangle|^2 < 1/3$ if the correct output is 1. **Hint**: We want to avoid summing over the state of the unmeasured "junk." Recall the trick we used to remove the junk produced by a reversible classical circuit.

If we want to simulate on a classical computer the quantum computation that solves the problem, then, it suffices to estimate the single matrix element $|\langle 0|\boldsymbol{U}|0\rangle|$ to reasonable accuracy.

Now recall that we saw in Exercise 3.3 that the quantum gate set $\{\boldsymbol{H}, \Lambda^2(\boldsymbol{X})\}$, where $\boldsymbol{H}$ denotes the Hadamard gate and $\Lambda^2(\boldsymbol{X})$ is the Toffoli gate, is universal for quantum computation. The Toffoli gate is classical, but the Hadamard gate takes the computational basis states of a qubit to superpositions of basis states:

$$\boldsymbol{H} : |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} (-1)^{xy}|y\rangle \ . \tag{27}$$

Note that, since $\boldsymbol{H}^2 = I$, we are free to insert a pair of Hadamard gates acting on all qubits following each Toffoli gate, without changing anything.

Suppose that $\boldsymbol{U}$ is expressed as a circuit constructed from Toffoli gates and Hadamard gates, where the circuit contains h Hadamard gates, which we label by $i = 0, 1, 2, \ldots, h-1$. By inserting the partition of unity $I = \sum_{x_i\in\{0,1\}} |x_i\rangle\langle x_i|$ following each Hadamard gate, we may write the matrix element $\langle 0|\boldsymbol{U}|0\rangle$ as a sum of 2^h terms, with each term arising from a "computational path" indexed by the bit string $x = x_{h-1}x_{h-2}\ldots x_1x_0$. Each term has absolute value $2^{-h/2}$, arising from the factor $2^{-1/2}$ that accompanies each Hadamard gate, and a phase ± 1 depending on x that counts modulo 2 the number of Hadamard gates for which the input and output both have the value 1. Note that each output bit from every Toffoli gate, and therefore the input to each Hadamard gate, can be expressed as a polynomial in the $\{x_i\}$ that depends on the circuit.

b) Show that, if $\boldsymbol{U}$ is a unitary transformation constructed from a circuit of size L, built from Toffoli and Hadamard gates, then

> there is a polynomial function $\phi(x)$ of $h \leq 3L$ binary variables $\{x_{h-1}, x_{h-2}, \ldots x_1, x_0\}$ such that
>
> $$\langle 0|\boldsymbol{U}|0\rangle = \frac{1}{\sqrt{2^h}}\left(N_0 - N_1\right) \ , \tag{28}$$
>
> where N_0 is the number of values of x for which $\phi(x) = 0$ (mod 2) and N_1 is the number of values of x for which $\phi(x) = 1$ (mod 2). Furthermore, $\phi(x)$ is of degree at most three, is a sum of at most $2h$ monomials, and can be computed efficiently from a description of the circuit.

Thus, if the circuit is polynomial size, the function $\phi(x)$ can be evaluated efficiently for any of the 2^h possible values of its input x. The difference $N_0 - N_1$ is at most $\sqrt{2^h}$, and we may simulate the quantum circuit (solving a problem in **BQP** by distinguishing $|\langle 0|U|0\rangle|^2 \geq 2/3$ from $|\langle 0|U|0\rangle|^2 < 1/3$) by estimating $N_0 - N_1$ to sufficient accuracy.

Up until now, the computational problems we have encountered have typically involved determining whether a solution to an equation exists, exhibiting a solution, or verifying the correctness of a solution. Here we have encountered a problem that appears to be intrinsically harder: *counting* (approximately) the number of solutions.

We could attempt to do the counting by a randomized computation, estimating $N_0 - N_1$ by evaluating $\phi(x)$ for a sample of randomly selected values of x. Unfortunately, the number of inputs mapped to 0 and to 1 are nearly in balance, which makes it hard to estimate $N_0 - N_1$. We can think of $\phi(x)$ as a coin with an exponentially small bias; determining the basis to reasonable accuracy requires an exponentially large number of trials.

But the task of simulating **BQP** within the class **PP** is easier than that — it is enough to be able to distinguish $|\langle 0|\boldsymbol{U}|0\rangle|^2 \geq 2/3$ from $|\langle 0|\boldsymbol{U}|0\rangle|^2 < 1/3$ with success probability $1/2+\delta$ as long as δ is positive, even if it is exponentially small. This crude estimate can be achieved with a polynomial number of trials. Therefore, $\mathbf{BQP} \subseteq \mathbf{PP}$.

The class **PP** is certainly contained in **PSPACE**, because we can determine the probability of acceptance for a randomized computation by simulating all of the possible computational paths that occur for all possible outcomes of the coin flips. There may be an exponentially large number of paths, but we can run through the paths one at a time, while maintaining a count of how many of the computations have accepted. This can be done with polynomial space.