Ph 219/CS 219

Exercises
Due: Friday 20 October 2006

1.1 How far apart are two quantum states?

Consider two quantum states described by density operators ρ and $\tilde{\rho}$ in an N-dimensional Hilbert space, and consider the complete orthogonal measurement $\{E_a, a = 1, 2, 3, \ldots N\}$, where the E_a's are one-dimensional projectors satisfying

$$\sum_{a=1}^{N} E_a = I \ . \tag{1}$$

When the measurement is performed, outcome a occurs with probability $p_a = \text{tr}\ \rho E_a$ if the state is ρ and with probability $\tilde{p}_a = \text{tr}\ \tilde{\rho} E_a$ if the state is $\tilde{\rho}$.

The *Kolmogorov distance* (or L^1 *distance*) between the two probability distributions is defined as

$$d(p, \tilde{p}) \equiv \|p - \tilde{p}\|_1 \equiv \frac{1}{2} \sum_{a=1}^{N} |p_a - \tilde{p}_a| \ ; \tag{2}$$

this distance is zero if the two distributions are identical, and attains its maximum value one if the two distributions have support on disjoint sets.

a) Show that

$$d(p, \tilde{p}) \leq \frac{1}{2} \sum_{i=1}^{N} |\lambda_i| \tag{3}$$

where the λ_i's are the eigenvalues of the Hermitian operator $\rho - \tilde{\rho}$. [**Hint**: Working in the basis in which $\rho - \tilde{\rho}$ is diagonal, find an expression for $|p_a - \tilde{p}_a|$, and then find an upper bound on $|p_a - \tilde{p}_a|$. Finally, use the completeness property eq. (1) to bound $d(p, \tilde{p})$.]

b) Find a choice for the orthogonal projector $\{E_a\}$ that saturates the upper bound eq. (3).

Define a distance $d(\rho,\tilde{\rho})$ between density operators as the maximal Kolmogorov distance between the corresponding probability distributions that can be achieved by any orthogonal measurement. From the results of (a) and (b), we have found that

$$d(\rho,\tilde{\rho}) = \frac{1}{2}\sum_{i=1}^{N} |\lambda_i| \ . \tag{4}$$

c) The *trace norm* $\|A\|_{\rm tr}$ of an operator A is defined as

$$\|A\|_{\rm tr} \equiv \mathrm{tr}\left[(AA^\dagger)^{1/2}\right] \ . \tag{5}$$

How can the distance $d(\rho,\tilde{\rho})$ be expressed as the trace norm of an operator?

Now suppose that the states ρ and $\tilde{\rho}$ are pure states $\rho = |\psi\rangle\langle\psi|$ and $\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$. If we adopt a suitable basis in the space spanned by the two vectors, these states can be expressed as

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \ , \quad |\tilde{\psi}\rangle = \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \ , \tag{6}$$

where $\langle\psi|\tilde{\psi}\rangle = \sin\theta$.

d) Express the distance $d(\rho,\tilde{\rho})$ in terms of the angle θ.

e) Express $\|\,|\psi\rangle - |\tilde{\psi}\rangle\|^2$ (where $\|\cdot\|$ denotes the Hilbert space norm) in terms of θ, and by comparing with the result of (d), derive the bound

$$d(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|) \le \|\,|\psi\rangle - |\tilde{\psi}\rangle\| \ . \tag{7}$$

f) Bob thinks that the norm $\|\,|\psi\rangle - |\tilde{\psi}\rangle\|$ should be a good measure of the distinguishability of the pure quantum states ρ and $\tilde{\rho}$. Explain why Bob is wrong. [**Hint**: Remember that quantum states are *rays*.]

1.2 The power of noncontextuality

We may regard a quantum state as an assignment of probabilities to projection operators. That is, according to Born's rule, if ρ is a density operator and E is a projector, then $p(E) = \mathrm{tr}\,(\rho E)$ is the probability that the outcome E occurs, if E is one of a complete set of orthogonal projectors associated with a particular quantum measurement. A notable feature of this rule is that the assignment of a probability $p(E)$ to E is *noncontextual*. This means that, while of course the probability $p(E)$ depends on the state ρ, it does not depend on how we choose the rest of the projectors that complete the orthogonal set containing E.

In a *hidden variable theory*, the probabilistic description of quantum measurement is derived from a more fundamental and complete deterministic description. The outcome of a measurement could be perfectly predicted if the values of the hidden variables were precisely known – then the probability $p(E)$ could take only the values 0 and 1. The standard probabilistic predictions of quantum theory arise when we average over the unknown values of the hidden variables. The purpose of this exercise is to show that such deterministic assignments conflict with noncontextuality. Thus a hidden variable theory, if it is to agree with the predictions of quantum theory after averaging, must be contextual.

Let $\{I, X, Y, Z\}$ denote the single-qubit observables

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\ Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \end{aligned} \tag{8}$$

and consider the nine two-qubit observables:

$$\begin{matrix} X\otimes I & I\otimes X & X\otimes X \\ I\otimes Y & Y\otimes I & Y\otimes Y \\ X\otimes Y & Y\otimes X & Z\otimes Z \end{matrix} . \tag{9}$$

The three observables in each row and in each column are mutually commuting, and so can be simultaneously diagonalized. In fact the simultaneous eigenstates of any two operators in a row or column (the third operator is not independent of the other two) are a complete basis for the four-dimensional Hilbert space of the two qubits. Thus we can regard the array eq. (9) as a way of presenting six different ways to choose a complete set of one-dimensional projectors for two qubits.

Each of these observables has eigenvalues ± 1, so that in a deterministic and noncontextual model of measurement (for a fixed value of the hidden variables), each can be assigned a definite value, either $+1$ or -1.

a) Any noncontextual deterministic assignment has to be consistent with the multiplicative structure of the observables. For example, the product of the three observables in the top row is the identity $I \otimes I$. Therefore, if we assign a value ± 1 to each operator, the number of -1's assigned to the first row must be even. Compute the product of the three observables in each row and each column to find the corresponding constraints.

b) Show that there is no way to satisfy all six constraints simultaneously.

Thus a deterministic and noncontextual assignment does not exist.

1.3 Schmidt-decomposable states.

We saw in class that any vector in a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed in the *Schmidt form*: Given the vector $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are both N-dimensional, we can choose orthonormal bases $\{|i\rangle_A\}$ for $\mathcal{H}_A$ and $\{|i\rangle_B\}$ for $\mathcal{H}_B$ so that

$$|\psi\rangle_{AB} = \sum_{i=1}^{N} \sqrt{\lambda_i}\ |i\rangle_A \otimes |i\rangle_B \ , \tag{10}$$

where the λ_i's are real and nonnegative. (We're not assuming here that the vector has unit norm, so the sum of the λ_i's is not constrained.) Eq. (10) is called the *Schmidt decomposition* of the vector $|\psi\rangle_{AB}$. Of course, the bases in which the vector has the Schmidt form depend on which vector $|\psi\rangle_{AB}$ is being decomposed.

A unitary transformation acting on $\mathcal{H}_{AB}$ is called a *local unitary* if it is a tensor product $U_A \otimes U_B$, where U_A, U_B are unitary transformations acting on $\mathcal{H}_A$, $\mathcal{H}_B$ respectively. The word "local" is used because if the two parts A and B of the system are widely separated from one another, so that Alice can access only part A and Bob can access only part B, then Alice and Bob can apply this transformation by each acting locally on her or his part.

a) Now suppose that Alice and Bob choose standard fixed bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ for their respective Hilbert spaces, and are presented with a vector $|\psi_{AB}\rangle$ that is not necessarily in the Schmidt form when expressed in the standard bases. Show that there is a local unitary $U_A \otimes U_B$ that Alice and Bob can apply so that the resulting vector

$$|\psi\rangle'_{AB} = U_A \otimes U_B |\psi\rangle_{AB} \tag{11}$$

does have the form eq. (10) when expressed in the standard bases.

b) Let's verify that the result of (*a*) makes sense from the point of view of parameter counting. For a *generic* vector in the Schmidt form, all λ_i's are nonvanishing and no two λ_i's are equal. Consider the *orbit* that is generated by letting arbitrary local unitaries act on one fixed generic vector in the Schmidt form. What is the dimension of the orbit, that is, how many real parameters are needed to specify one particular vector on the orbit? [**Hint**: To do the counting, consider the local unitaries that differ infinitesimally from the identity $I_A \otimes I_B$. Choose a basis for these, and count the number of independent linear combinations of the basis elements that annihilate the Schmidt-decomposed vector.] Compare the dimension of the orbit to the (real) dimension of $\mathcal{H}_{AB}$, and check the consistency with the number of free parameters in eq. (10).

A vector $|\psi\rangle_{A_1 \ldots A_r}$ in a Hilbert space $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_r}$ with r parts is said to be *Schmidt decomposable* if it is possible to choose orthonormal bases for $\mathcal{H}_{A_1}, \ldots \mathcal{H}_{A_r}$ such that vector can be expressed as

$$|\psi\rangle_{A_1 \ldots A_r} = \sum_i \sqrt{\lambda_i}\; |i\rangle_{A_1} \otimes |i\rangle_{A_2} \otimes \cdots \otimes |i\rangle_{A_r} \;. \tag{12}$$

Though every vector in a bipartite Hilbert space is Schmidt decomposable, this isn't true for vectors in Hilbert spaces with three or more parts.

c) Consider a generic Schmidt-decomposable vector in the tripartite Hilbert space of three qubits. Find the dimension of the orbit generated by local unitaries acting on this vector.

d) By considering the number of free parameters in the Schmidt form eq. (12), and the result of (*c*), find the (real) dimension of the space of Schmidt-decomposable vectors for three qubits. What is the real *codimension* of this space in the three-qubit Hilbert space $\mathbf{C}^8$?

1.4 Completeness of subsystem correlations

Consider a bipartite system AB. Suppose that many copies of the (not necessarily pure) state ρ_{AB} have been prepared. An observer Alice with access only to subsystem A can measure the expectation value of any observable of the form $M_A \otimes I_B$, while an observer Bob with access only to subsystem B can measure the expectation value of any observable of the form $I_A \otimes N_B$. As discussed in class, neither of these observers, working alone, can expect to learn very much about the state ρ_{AB}.

But now suppose that Alice and Bob can communicate, exchanging (classical) information about how their measurement outcomes are *correlated*. Thereby, they can jointly determine the expectation value of any observable of the form $M_A \otimes N_B$ (an observable whose eigenstates are separable direct products states of the form $|\varphi\rangle_A \otimes |\chi\rangle_B$).

The point of this exercise is to show that if Alice and Bob have complete knowledge of the nature of the correlations between subsystems A and B (know the expectation values of any tensor product observable $M_A \otimes N_B$), then in fact they know everything about the bipartite state ρ_{AB} – there will be no surprises when they measure entangled observables, those whose eigenstates are entangled states.

a) Let $\{M_a\,,\ a = 1, 2, \ldots, N^2\}$ denote a set of N^2 linearly independent self-adjoint operators acting on a Hilbert space $\mathcal{H}$ of dimension N. Show that if ρ is a density operator acting on $\mathcal{H}$, and $\mathrm{tr}\,(\rho\ M_a)$ is known for each a, then $\langle\varphi|\rho|\varphi\rangle$ is known for any vector $|\varphi\rangle$ in $\mathcal{H}$.

b) Show that if $\langle\varphi|\rho|\varphi\rangle$ is known for each vector $|\varphi\rangle$, then ρ is completely known.

c) Show that if $\{M_a\}$ denotes a basis for self-adjoint operators on $\mathcal{H}_A$, and $\{N_b\}$ denotes a basis for self-adjoint operators on $\mathcal{H}_B$, then $\{M_a \otimes N_b\}$ is a basis for the self-adjoint operators on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Remark: It follows from (c) alone that the correlations of the "local" observables determine the expectation values of all the observables. Parts (a) and (b) serve to establish that ρ is completely determined by the expectation values of a complete set of observables.

d) State and prove the result corresponding to (c) that applies to a multipartite system with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$.

e) Discuss how the world would be different if quantum states resided in a real Hilbert space rather than a complex Hilbert space. Consider, in particular, whether (c) is true for *symmetric* operators acting on a real vector space.