Ph 219a/CS 219a

Exercises
Due: Friday 2 December 2005

3.1 Two inequivalent types of tripartitite entangled pure states

Alice, Bob, and Charlie share a *GHZ state* of three qubits:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \ , \tag{1}$$

but they wish they had a *W state* instead:

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \ . \tag{2}$$

They are able to communicate classically as much as they like, but unfortunately, they are unable to exchange quantum information with one another. Can they transform $|\mathrm{GHZ}\rangle$ to $|W\rangle$ using only local operations and classical communication (LOCC)?

Alice, Bob, and Charlie realize that there might not exist any LOCC protocol that is guaranteed to succeed in transforming $|\mathrm{GHZ}\rangle$ into $|W\rangle$ — they will settle for a small, but nonzero, probability of success. Then we say the protocol is SLOCC (for "stochastic LOCC"). Is such a SLOCC protocol possible?

Let us say that two three-qubit states are SLOCC equivalent if there is a SLOCC protocol that transforms one to the other. The objective of this problem is to show that $|\mathrm{GHZ}\rangle$ and $|W\rangle$ are not SLOCC equivalent. In this sense the GHZ state and the W state are representatives of two inequivalent classes of three-qubit pure states that are entangled in essentially different ways.

Let $|\varphi\rangle$ and $|\psi\rangle$ be two pure states of the three-qubit system ABC, both with the property that all of the single-qubit marginal density operators have full rank — that is, each of ρ_A, ρ_B, ρ_C has two eigenvectors with nonzero eigenvalues. Suppose that Alice, Bob, and Charlie initially share $|\varphi\rangle$, that each party performs a local POVM on his/her system, and that each broadcasts the results of his measurement to the other parties. Suppose that, with some nonzero probability of success, this measurement achieves the transformation $|\varphi\rangle \mapsto |\psi\rangle$.

a) Show that there exist invertible 2×2 matrices A, B, and C such that

$$|\psi\rangle = (A \otimes B \otimes C)|\varphi\rangle \ . \tag{3}$$

b) Suppose that $|\varphi\rangle$ can be expressed as a linear combination of two linearly independent product states of three qubits. Show that the same must be true for $|\psi\rangle$.

c) Suppose that $|\varphi\rangle$ can be expressed as a linear combination of two linearly independent three-qubit product states (and that ρ_A, ρ_B, ρ_C have full rank). Show that the *range* of ρ_{BC} contains two linearly independent (two-qubit) product states (where $\rho_{BC} = \mathrm{tr}_A(|\varphi\rangle\langle\varphi|)$).

d) For both $|\mathrm{GHZ}\rangle$ and $|W\rangle$, calculate the marginal density operators ρ_A, ρ_B, ρ_C, ρ_{AB}, ρ_{AC}, ρ_{BC}.

e) Show that $|\mathrm{GHZ}\rangle$ and $|W\rangle$ are not SLOCC equivalent.

Remark: We may say that two n-qubit pure quantum states $|\varphi\rangle$ and $|\psi\rangle$ are SLOCC equivalent if there is a SLOCC protocol that transforms one to the other. The arguments above show that $|\varphi\rangle$ and $|\psi\rangle$ are SLOCC equivalent if there is a tensor product of invertible matrices $A = \otimes_i^n A_i$ such that $|\psi\rangle = A|\varphi\rangle$. For $n = 2$ there is just one nontrivial SLOCC equivalence class (product states are the trivial class). For $n = 3$, aside from the trivial cases (products of a one-qubit state with a two-qubit state), there are two classes — every nontrivial state can be transformed by SLOCC into either a GHZ state of a W state. The GHZ class is generic, while the W class has measure zero. For more qubits, the classification becomes a lot more complicated.

3.2 Coherent classical communication

We saw that the tasks realized by superdense coding (I called it "dense coding" in class, but I will use the more standard locution "superdense coding" in this problem, so I can abbreviate it by SD) and by teleportation (TP) can be succinctly expressed as *resource inequalities*:

$$\begin{aligned} [q \to q] + [qq] &\geq 2[c \to c] \qquad (\mathrm{SD})\ , \\ 2[c \to c] + [qq] &\geq [q \to q] \qquad (\mathrm{TP})\ . \end{aligned} \tag{4}$$

Here $[c \to c]$ denotes one classical bit (cbit) sent from Alice to Bob, $[q \to q]$ denotes one qubit sent from Alice to Bob, and $[qq]$ denotes one *ebit* — a maximally entangled pair of qubits shared by Alice and Bob. The meaning of the inequality is that the input resources on the left can be converted into the output resources on the right.

These inequalities are strict, in the sense that the resource conversions are irreversible — there is no protocol corresponding to SD or TP "running backwards." It turns out that there is a natural way to formulate versions of SD and TP that are reversible in the sense that the resource inequality can be replaced by an equality, but to do so we must replace classical communication with a stronger resource: *coherent classical communication.*

The three types of communication that we wish to consider can be realized as isometries (inner-product preserving linear maps). Sending a qubit from Alice to Bob can be expressed as

$$[q \to q] : |x\rangle_A \to |x\rangle_B \ , \tag{5}$$

where $x \in \{0, 1\}$. Sending a cbit from Alice to Bob can be expressed as

$$[c \to c] : |x\rangle_A \to |x\rangle_B \otimes |x\rangle_E \ ; \tag{6}$$

here E denotes an environment that cannot be accessed by either Alice or Bob. The communication is classical because a quantum signal that Alice attempts to send to Bob decoheres in the basis $\{|0\rangle, |1\rangle\}$. Sending a *cobit* (a unit of coherent classical communication) from Alice to Bob can be expressed as

$$[q \to qq] : |x\rangle_A \to |x\rangle_A \otimes |x\rangle_B \ ; \tag{7}$$

this is somewhat like classical communication, except that Alice maintains control of the "environment."

a) Show that

$$[q \to q] \geq [q \to qq] \geq [c \to c] \ , \quad \text{and} \quad [q \to qq] \geq [qq] \ . \tag{8}$$

(That is, explain how to use what is on the left side of each inequality to achieve what is on the right side.)

b) Show that

$$[q \to q] + [qq] \geq 2[q \to qq] \ . \tag{9}$$

Hint: Use a coherent version of superdense coding. That is, Alice has two qubits in the state $|xy\rangle_A$ and she shares a Bell pair with Bob. She can achieve $|xy\rangle_A \to |xy\rangle_A \otimes |xy\rangle_B$ by applying an appropriate unitary transformation to her two qubits and her half of the entangled pair, and then sending her half of the Bell pair to Bob.

c) Show that

$$2[q \to qq] + [qq] \geq [q \to q] + 2[qq] \ . \tag{10}$$

Hint: Use teleportation, but where Alice performs an appropriate unitary transformation instead of a Bell measurement, and where the classical communication from Alice to Bob is coherent. (Be sure to verify that Alice and Bob wind up with 2 ebits at the end of the protocol.)

Remark: We see that coherent teleportation actually creates more entanglement than it consumes. We can express the results of (b) and (c) by saying

$$2[q \to qq] = [q \to q] + [qq] \ , \tag{11}$$

but where the the equality indicates that the resource conversion could be *catalytic* — a resource might need to be borrowed to activate the process, but this borrowed resource can be returned when the protocol is completed.

3.3 Bipartite mixed-state entanglement

We say that mixed state ρ_{AB} of the bipartite state AB is *separable* if it can be realized as an ensemble of product states:

$$\rho_{AB} = \sum_{i,j} p_{ij} \ \rho_{A,i} \otimes \rho_{B,j} \ , \tag{12}$$

where $\rho_{A,i}$ is a density operator on A, $\rho_{B,j}$ is a density operator on B, and the $\{p_{ij}\}$ are nonnegative real numbers such that $\sum_{ij} p_{ij} = 1$.

a) Show that ρ_{AB} is separable if and only if it can be expressed as

$$\rho_{AB} = \sum_{i} p_i \ \rho_{A,i} \otimes \rho_{B,i} \ , \tag{13}$$

where each $\rho_{A,i}$ and $\rho_{B,i}$ is a *pure* state, and the $\{p_i\}$ are nonnegative real numbers such that $\sum_i p_i = 1$.

Recall that for any choice of an orthonormal basis $\{|i\rangle\}$ on system B, the *transpose* T acts according to

$$T : |i\rangle\langle j| \mapsto |j\rangle\langle i| \ . \tag{14}$$

We have seen that, although T is a positive map, it is not completely positive — that is, the "partial transpose" map $I \otimes T$ acting on AB is not positive if $\dim A = \dim B$. Specifically, if $|\Phi\rangle$ is the maximally entangled state

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle_A \otimes |i\rangle_B \ , \tag{15}$$

then

$$I \otimes T : |\Phi\rangle\langle\Phi| \mapsto \frac{1}{N}\,(\mathrm{SWAP}) \ , \tag{16}$$

where

$$\mathrm{SWAP}\,(|i\rangle_A \otimes |j\rangle_B) = |j\rangle_A \otimes |i\rangle_B \ . \tag{17}$$

Any antisymmetric state on AB is an eigenstate of SWAP with eigenvalue -1; therefore SWAP is not a positive operator.

b) Show that if the bipartite density operator ρ_{AB} is separable, then $I \otimes T(\rho_{AB})$ is also a density operator (it is nonnegative and has unit trace). **Remark**: This is called the *positive partial transpose* (PPT) criterion for separability. In general, positivity of the partial transpose is a necessary condition for separability, but it is not necessarily sufficient. However, for states of two qubits it is known to be both necessary and sufficient.

Consider the mixed state of two qubits

$$\rho_F = F|\phi^+\rangle\langle\phi^+| + \frac{1-F}{3}\left(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + \psi^-\rangle\langle\psi^-|\right) \ ; \tag{18}$$

here $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$ are the Bell states

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \ , \quad |\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \ . \tag{19}$$

The density operator ρ_F is called the *Werner state* with fidelity F. Note that ρ_F is the state that results if the depolarizing channel with error probability $1-F$ acts on half of the maximally entangled state $|\phi^+\rangle$.

c) Show that ρ_F can be rewritten as

$$\rho_F = \lambda|\phi^+\rangle\langle\phi^+| + \frac{1-\lambda}{4}(I\otimes I) \ , \tag{20}$$

and express λ is terms of F.

d) Using the PPT criterion (which is necessary and sufficient for separability in the case of two qubits), find the value $F_{\rm PPT}$ such that ρ_F is entangled for $F_{\rm PPT} < F \le 1$.

e) Suppose that Alice and Bob share the state ρ_F. Alice measures one of two observables, a or a', and Bob measures one of two observables, b or b'. All four observables have two possible outcomes, ± 1. Find the value $F_{\rm CHSH}$ such that the state ρ_F violates the CHSH inequality for $F_{\rm CHSH} < F \le 1$. That is, for F in this range Alice and Bob can choose observables whose correlator satisfies

$$\langle ab + a'b + ab' - a'b'\rangle > 2 \ . \tag{21}$$

Remark: Note that $F_{\rm PPT} < F_{\rm CHSH}$. Curiously, and in contrast to the situation for bipartite pure states, *entangled* is not equivalent to *Bell-inequality violating* for bipartite mixed states. In fact, there is a value $F_{\rm local}$ such that for $0 \le F \le F_{\rm local}$, a local hidden variable theory can be constructed that accounts correctly for the statistics of all possible projective measurements that can be performed by Alice and Bob on the state ρ_F. The exact value of $F_{\rm local}$ is still unknown, but it is known that $F_{\rm PPT} < .7446 \le F_{\rm local} \le F_{\rm CHSH}$.

3.4 Separability and majorization

The hallmark of entanglement is that in an entangled state the whole is less random than its parts. But in a separable state the correlations are essentially classical and so are expected to adhere to the classical principle that the parts are less disordered than the whole. The objective of this problem is to make this expectation precise by showing that if the bipartite (mixed) state ρ_{AB} is separable, then

$$\lambda(\rho_{AB}) \prec \lambda(\rho_A) \ , \quad \lambda(\rho_{AB}) \prec \lambda(\rho_B) \ . \tag{22}$$

Here $\lambda(\rho)$ denotes the vector of eigenvalues of ρ, and $\prec$ denotes majorization.

A separable state can be realized as an ensemble of pure product states, so that if ρ_{AB} is separable, it may be expressed as

$$\rho_{AB} = \sum_a p_a \, |\psi_a\rangle\langle\psi_a| \otimes |\varphi_a\rangle\langle\varphi_a| \ . \tag{23}$$

We can also diagonalize ρ_{AB}, expressing it as

$$\rho_{AB} = \sum_j r_j |e_j\rangle\langle e_j| \ , \tag{24}$$

where $\{|e_j\rangle\}$ denotes an orthonormal basis for AB; then by the HJW theorem, there is a unitary matrix V such that

$$\sqrt{r_j}|e_j\rangle - \sum_a V_{ja}\sqrt{p_a}|\psi_a\rangle \otimes |\varphi_a\rangle \ . \tag{25}$$

Also note that ρ_A can be diagonalized, so that

$$\rho_A = \sum_a p_a|\psi_a\rangle\langle\psi_a| = \sum_\mu s_\mu|f_\mu\rangle\langle f_\mu| \ ; \tag{26}$$

here $\{|f_\mu\rangle\}$ denotes an orthonormal basis for A, and by the HJW theorem, there is a unitary matrix U such that

$$\sqrt{p_a}|\psi_a\rangle = \sum_\mu U_{a\mu}\sqrt{s_\mu}|f_\mu\rangle \ . \tag{27}$$

Now show that there is a doubly stochastic matrix D such that

$$r_j = \sum_\mu D_{j\mu} s_\mu \ . \tag{28}$$

That is, you must check that the entries of $D_{j\mu}$ are real and nonnegative, and that $\sum_j D_{j\mu} = 1 = \sum_\mu D_{j\mu}$. Thus we conclude that $\lambda(\rho_{AB}) \prec \lambda(\rho_A)$. Just by interchanging A and B, the same argument also shows that $\lambda(\rho_{AB}) \prec \lambda(\rho_B)$.

Remark: Note that it follows from the Schur concavity of Shannon entropy that, if ρ_{AB} is separable, then the von Neumann entropy has the properties $H(AB) \geq H(A)$ and $H(AB) \geq H(B)$. Thus, for separable states, conditional entropy is nonnegative: $H(A|B) = H(AB) - H(B) \geq 0$ and $H(B|A) = H(AB) - H(A) \geq 0$. In contrast, if the state of AB is an entangled pure state, then $H(AB) = 0$ and $H(B|A) = H(A|B) < 0$.