Ph 219c/CS 219c

Exercises
Due: Friday 16 February 2007

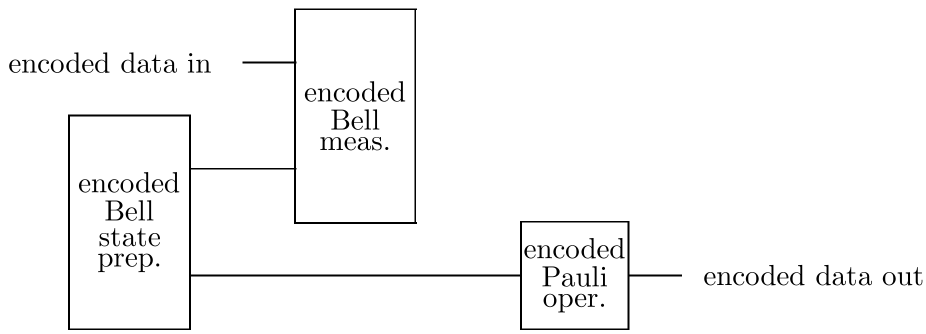
6.1 Fault-tolerant error correction by teleportation

a) A destructive fault-tolerant encoded Bell measurement can be done *transversally* for any stabilizer code. Consider a stabilizer code with length n and k encoded qubits. Denote the $n-k$ stabilizer generators by $\{M_i,\ i = 1, 2, 3, \ldots n-k\}$, and denote the encoded Pauli operators by $\{\bar{X}_a, \bar{Y}_a, \bar{Z}_a,\ a = 1, 2, 3, \ldots k\}$. If two code blocks A and B are both encoded using this code, then by an encoded Bell measurement of encoded qubit a performed on these two blocks we mean a simultaneous measurement of the two commuting operators $\bar{X}_{A,a} \otimes \bar{X}_{B,a}$ and $\bar{Z}_{A,a} \otimes \bar{Z}_{B,a}$.

Denote the Pauli operators acting on the n qubits in the code block by $\{X_\alpha, Y_\alpha, Z_\alpha,\ \alpha = 1, 2, 3, \ldots n\}$. Suppose that, instead of the encoded Bell measurement, a Bell measurement is performed *bitwise* — that is $X_{A,\alpha} \otimes X_{B,\alpha}$ and $Z_{A,\alpha} \otimes Z_{B,\alpha}$ are measured for all $\alpha = 1, 2, 3, \ldots n$. (Each of these two-qubit Bell measurements can be realized by performing a CNOT gate followed by an X measurement on the CNOT's control qubit and a Z measurement on its target qubit.) Explain how the results of the bitwise Bell measurements can be processed to infer the outcomes of the ideal encoded Bell measurements for all $a = 1, 2, 3, \ldots k$.

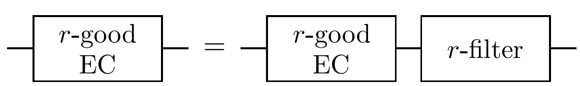
b) Now suppose that some of the bitwise Bell measurements are faulty (the recorded value of $X_{A,\alpha} \otimes X_{B,\alpha}$, or $Z_{A,\alpha} \otimes Z_{B,\alpha}$, or both differs from the ideal outcome). Suppose that the stabilizer code has distance $d = 2t + 1$ so that it can correct t errors. Explain how, if at most t of the n bitwise Bell measurements are faulty, the outcomes of the ideal Bell measurements can still be inferred correctly.

c) A particularly efficient scheme for fault-tolerant error correction, which can be carried out for any stabilizer code, is *error correction by teleportation*:

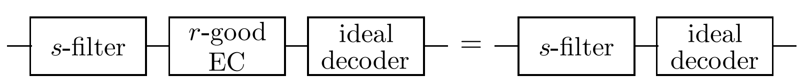


First the encoded Bell state $\bigotimes_a |\bar{\phi}^+_a\rangle_{BC}$ is prepared and verified, then encoded Bell measurement is performed on block B and the incoming data block A, and finally an encoded Pauli operator is applied to complete the teleportation of the incoming encoded state.

Suppose that the preparation and verification circuit for the encoded Bell state has the property that if the circuit contains no more than r faults, then each of its output blocks has no more than r errors. Show that in that case the circuit for error correction by teleportation is fault tolerant — that is it has the two properties:



$(r \leq t)$



$(r + s \leq t)$

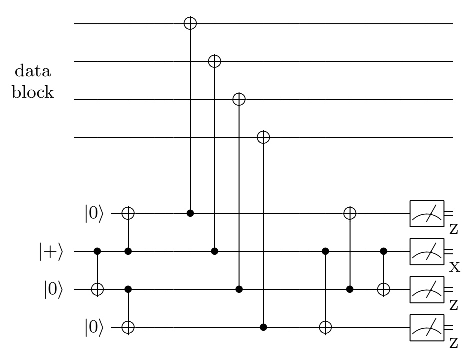
Here we use "r-good" to indicate an error correction with at most r faults, and the "s-filter" is the orthogonal projection onto the space spanned by all states that can be obtained by acting on a codeword with a Pauli operator of weight no larger than s.

6.2 Fault-tolerant error correction without ancilla verification

In one version of fault-tolerant error correction discussed in class, the stabilizer generators of a stabilizer code are measured by the cat state method: first a cat state is prepared and verified, and the cat state is used for the measurement only if the verification step confirms that the preparation did not fail badly. In this scheme, one must wait for the outcome of the measurement performed in the verification step before knowing whether the cat state will be accepted. Having to wait for the measurement to be completed might be a big disadvantage in a setting where it takes much longer to do a measurement than the time required to execute a gate.

Fortunately, though, there is an alternative procedure where waiting is not necessary. In this procedure, instead of being verified before use, the ancilla is measured after use. Though an encoded error might propagate from a badly damaged ancilla to the data, it will be known (later) when this has occured. The propagation of encoded Pauli operators through subsequent (Clifford group) gates in the circuit can be simulated efficiently on a classical computer, and with this information we can infer what the outcomes of the final measurements at the end of the computation would have been if the cat state had been prepared properly.

For example, for the [[7,1,3]] quantum code that corrects one error, consider measuring one of the weight-four X-type stabilizer generators, using the circuit shown here:



The four-qubit cat state is encoded with four qubit preparations and three CNOT gates, the ancilla interacts with the data block via four CNOT gates, the ancilla is decoded with three CNOTs, and then all four ancilla qubits are measured as shown.

a) To extract the eigenvalue of the stabilizer generator, we are to determine the eigenvalue of $X_1X_2X_3X_4$, acting on the cat state immediately after the cat state interacts with the data block. (Here e.g. X_1 denotes X acting on the first qubit of the cat state, and the tensor product symbol has been suppressed.). Show that it there are no faults, then the desired measurement is achieved by the final X_2 measurement in the circuit shown. For this purpose, you may find it useful to recall the action by conjugation of the CNOT gate on the Pauli operators

$$\text{CNOT}: \quad XI \rightarrow XX\ , \quad IX \rightarrow IX\ , \quad ZI \rightarrow ZI\ , \quad IZ \rightarrow ZZ\ ,$$

where the first qubit is the control qubit and the second qubit is the target qubit of the CNOT.

b) If there are no faults, what will the measurements of Z_1, Z_3, and Z_4 yield? (Write down the complete stabilizer of the cat state, and consider how each stabilizer generator propagates through the CNOT gates.)

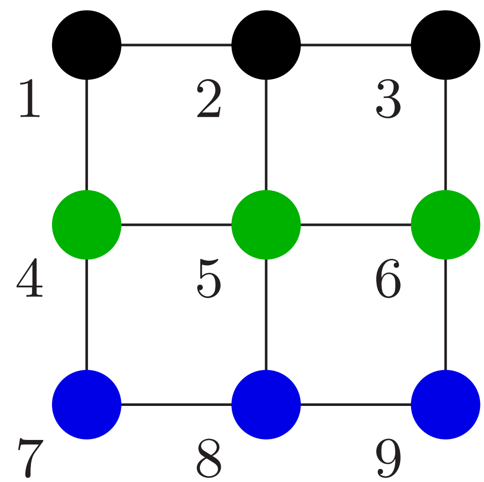
c) Faults in the cat state encoder might produce X errors that feed to the data via the following CNOT gates. In particular, a weight-two X error that propagates to the data will produce an encoded $\bar{X}$ error. What possible weight-two X errors in the cat state could be produced by a single fault in the cat state encoder? Explain how these weight-two X errors might arise.

d) If the cat state has a weight-one X error after interacting with the data, and there are no subsequent faults in the circuit, what are the possible outcomes of the Z_1, Z_3, and Z_4 measurements? If the cat state has a weight-two X error that could have arisen from a single fault in the cat state encoder, what are the possible outcomes of the Z_1, Z_3, and Z_4 measurements? Do these measurements distinguish weight-two X errors from X errors of weight one or zero?

e) If the only fault in the circuit occurs during the decoding and measurement of the cat state, what are the possible outcomes for the measurements of Z_1, Z_3, and Z_4? Finally, is this syndrome extraction scheme fault tolerant? (Can a single fault cause an encoded error?)

6.3 Fault-tolerant error correction via gauge qubit measurement

Shor's [[9,1,3]] quantum code has a nice interpretation that can be appreciated by laying out the nine qubits on a 3×3 grid:



The encoded Pauli operator $\bar{X}$ can be chosen to be the tensor product of X's acting on all the qubits in a row, e.g., $\bar{X} = X_{\mathrm{row}-1} = X_1 X_2 X_3$ and the encoded Pauli operator $\bar{Z}$ can be chosen to be the tensor product of Z's acting on all the qubits in column, e.g., $\bar{Z} = Z_{\mathrm{col}-1} = Z_1 Z_4 Z_7$. Furthermore, the tensor product of X's on all the qubits in two rows commutes with $\bar{Z}$ and the tensor product of Z's on all the qubits in two columns commutes with $\bar{X}$. Hence we may take the stabilizer generators of the code to be

$$\begin{matrix} X_{\mathrm{row}-1} X_{\mathrm{row}-2} \ , & X_{\mathrm{row}-2} X_{\mathrm{row}-3} \ , \\ Z_{\mathrm{col}-1} Z_{\mathrm{col}-2} \ , & Z_{\mathrm{col}-2} Z_{\mathrm{col}-3} \ , \end{matrix} \tag{1}$$

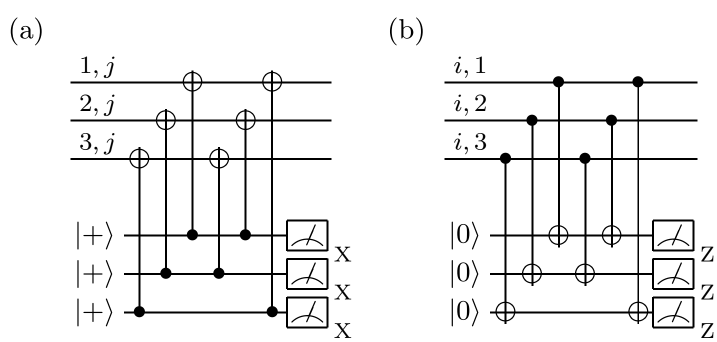
which are mutually commuting. Note that the encoded $\bar{X}$ may be taken to be any of $X_{\mathrm{row}-1}$, $X_{\mathrm{row}-2}$, or $X_{\mathrm{row}-3}$, as these differ by multiplication by an element of the stabilizer, and similarly, the encoded $\bar{Z}$ may be taken to be any of $Z_{\mathrm{col}-1}$, $Z_{\mathrm{col}-2}$, or $Z_{\mathrm{col}-3}$.

There is a (nonabelian) algebra of Pauli operators that commute with the encoded operations and with the stabilizer. This algebra includes products of neighboring X's in the same column (e.g. X_1X_4), products of neighboring Z's in the same row (e.g., Z_1Z_2), and all products of such operators. The Pauli operators in this algebra are harmless errors that preserve the stabilizer and have no effect on the encoded qubit. We will refer to the operators that commute with $\bar{X}$ and $\bar{Z}$, but that are not themselves elements of the stabilizer, as "gauge-qubit operators" (the terminology comes from an analogy with e.g. electrodynamics, where a "gauge transformation" has no effect on physical observables). A basis for the gauge-qubit operators is provided by, for example $\{X_1X_4, Z_1Z_3, X_2X_5, Z_2Z_3, X_4X_7, Z_7Z_9, X_5X_8, Z_8Z_9\}$.

From the perspective of fault tolerance, a particularly nice feature is that (even though the gauge qubit operators are not mutually commuting) the values of the outcomes of gauge qubit measurements can be used to infer the values of the eigenvalues of the stabilizer generators.

a) If gauge qubits can be measured without faults, explain how such measurements can be used to determine the error syndrome, and how the syndrome determines the appropriate recovery operation.

A procedure for measuring the weight-two gauge qubit operators in (a) column j and (b) row i is shown here:



E.g., for $j = 1$, the procedure (a) measures X_1X_4, X_4X_7, and X_1X_7, and for $i = 1$, the procedure (b) measures Z_1Z_2, Z_2Z_3, and Z_1Z_3. Though X_1X_7 is not independent of the other two observables in the first column, this third redundant measurement is needed to ensure fault tolerance. Similarly, in (b) the third measurement in each row ensures fault tolerance.

b) Suppose that the third redundant measurement in each row and column were omitted. How could a single faulty gate cause an encoded error? Show that the error correction procedure is fault tolerant (in the sense that the properties listed in Problem 6.1c are satisfied) when the redundant measurements are included.

c) How many locations (qubit preparations, qubit measurements, and CNOT gates) are contained in one complete cycle of fault-tolerant syndrome measurement?