Ph219/CS219 Problem Set 1

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October 20, 2006

1.1 How far apart are two quantum states?

(a) With $p_a = \mathrm{tr}(\rho E_a)$ and $\tilde{p}_a = \mathrm{tr}(\tilde{\rho} E_a)$,

$$d(p, \tilde{p}) = \frac{1}{2} \sum_{a=1}^{N} |p_a - \tilde{p}_a| = \frac{1}{2} \sum_{a=1}^{N} |\mathrm{tr}[(\rho - \tilde{\rho}) E_a]|.$$

$\rho - \tilde{\rho} = (\rho - \tilde{\rho})^\dagger$ is Hermitian, so it has a spectral decomposition (or diagonal representation):

$$\rho - \tilde{\rho} = \sum_{i=1}^{N} \lambda_i |i\rangle\langle i|, \qquad \lambda_i \in \mathbb{R},$$

where $\{|i\rangle\}$ are the eigenvectors and they form an orthonormal (ON) basis for the Hilbert space. Then,

$$\begin{aligned} d(p, \tilde{p}) &= \frac{1}{2} \sum_{a=1}^{N} \left| \mathrm{tr} \left(\sum_{i=1}^{N} \lambda_i |i\rangle\langle i| E_a \right) \right| \\ &= \frac{1}{2} \sum_{a=1}^{N} \left| \sum_{i=1}^{N} \lambda_i \langle i| E_a |i\rangle \right| \\ &\leq \frac{1}{2} \sum_{i=1}^{N} |\lambda_i| \sum_{a=1}^{N} |\langle i| E_a |i\rangle|. \end{aligned}$$

Now, $\langle i| E_a |i\rangle = \mathrm{tr}\left(|i\rangle\langle i| E_a\right) = \text{Prob.(outcome } a \text{ given state } |i\rangle\langle i|) \geq 0$, so $|\langle i| E_a |i\rangle| = \langle i| E_a |i\rangle$ and we have,

$$d(p, \tilde{p}) \leq \frac{1}{2} \sum_{i=1}^{N} |\lambda_i| \langle i| \underbrace{\left(\sum_{a=1}^{N} E_a \right)}_{=I \text{ by eq. (1)}} |i\rangle = \frac{1}{2} \sum_{i=1}^{N} |\lambda_i|.$$

(b) Define $E_i := |i\rangle\langle i|$ where $\{|i\rangle\}_{i=1}^{N}$ is the ON basis in which $\rho - \tilde{\rho}$ is diagonal. Notice that $\{E_i\}_{i=1}^{N}$ is complete ($\sum_{i=1}^{N} E_i = I$), one-dimensional,

orthogonal ($E_iE_j = \delta_{ij}E_i$), and saturates bound (3):

$$d(p,\tilde{p}) = \frac{1}{2}\sum_{i=1}^{N}\left|\sum_{j=1}^{N}\lambda_j\langle j|E_i|j\rangle\right| = \frac{1}{2}\sum_{i=1}^{N}\left|\sum_{j=1}^{N}\lambda_j\delta_{ij}\right| = \frac{1}{2}\sum_{i=1}^{N}|\lambda_i|.$$

Therefore, $\{E_i = |i\rangle\langle i|\}_{i=1}^{N}$ works.

(c) Using the given definition of the trace norm, we can compute

$$\|\rho-\tilde{\rho}\|_{tr} = \mathrm{tr}\left\{\left[(\rho-\tilde{\rho})^2\right]^{1/2}\right\} = \mathrm{tr}\left(\sum_{i=1}^{N}|\lambda_i|\,|i\rangle\langle i|\right) = \sum_{i=1}^{N}|\lambda_i| = 2d(\rho,\tilde{\rho}).$$

Therefore,

$$d(\rho,\tilde{\rho}) = \frac{1}{2}\|\rho-\tilde{\rho}\|_{tr}\,.$$

(d) To compute $d(\rho,\tilde{\rho})$, we need the eigenvalues of the Hermitian operator $\rho-\tilde{\rho}$:

$$\rho-\tilde{\rho} = |\psi\rangle\langle\psi| - |\tilde{\psi}\rangle\langle\tilde{\psi}| = \begin{pmatrix}\cos\theta & 0\\ 0 & -\cos\theta\end{pmatrix}.$$

$\rho-\tilde{\rho}$ is already diagonal in this basis, so we can read off the eigenvalues as $\pm\cos\theta$. Therefore,

$$d(\rho,\tilde{\rho}) = \frac{1}{2}\left(\,|\cos\theta| + |-\cos\theta|\,\right) = |\cos\theta|.$$

(e) The Hilbert space norm of $|\psi\rangle - |\tilde{\psi}\rangle$ is given by

$$\begin{aligned}\||\psi\rangle - |\tilde{\psi}\rangle\|^2 &= \left(\langle\psi| - \langle\tilde{\psi}|\right)\left(|\psi\rangle - |\tilde{\psi}\rangle\right)\\ &= \langle\psi|\psi\rangle - \left(\langle\psi|\tilde{\psi}\rangle + \langle\tilde{\psi}|\psi\rangle\right) + \langle\tilde{\psi}|\tilde{\psi}\rangle\\ &= 2(1-\sin\theta).\end{aligned}$$

Compare this with the result for $d\left(|\psi\rangle\langle\psi| - |\tilde{\psi}\rangle\langle\tilde{\psi}|\right)$ from part (c):

$$\begin{aligned}\left[d\left(|\psi\rangle\langle\psi| - |\tilde{\psi}\rangle\langle\tilde{\psi}|\right)\right]^2 &= \cos^2\theta\\ &= 1-\sin^2\theta\\ &= \underbrace{(1+\sin\theta)}_{\leq 2}(1-\sin\theta)\\ &\leq 2(1-\sin\theta) = \||\psi\rangle - |\tilde{\psi}\rangle\|^2\end{aligned}$$

$$\Rightarrow \qquad d\left(|\psi\rangle\langle\psi| - |\tilde{\psi}\rangle\langle\tilde{\psi}|\right) \leq \||\psi\rangle - |\tilde{\psi}\rangle\|$$

(f) Consider the states $|\psi\rangle$ and $|\tilde{\psi}\rangle$ with $\theta = \frac{3\pi}{2}$. It is easy to see that $|\tilde{\psi}\rangle = -|\psi\rangle$, with $\langle\psi|\psi\rangle = 1$, normalized. Now, recall that physical states are really rays, i.e. they are equivalence classes of state vectors differing by an overall phase. So, $|\psi\rangle$ and $|\tilde{\psi}\rangle$ should represent the same physical state, and hence should be indistinguishable from each other. However, if you compute the norm $\||\psi\rangle - |\tilde{\psi}\rangle\|$, you will find:

$$\||\psi\rangle - |\tilde{\psi}\rangle\| = \||\psi\rangle + |\psi\rangle\| = 2\||\psi\rangle\| = 2,$$

which is the maximum attainable value for normalized states. If $\||\psi\rangle - |\tilde{\psi}\rangle\|$ is a good measure of distinguishability, this would say that $|\psi\rangle$ and $|\tilde{\psi}\rangle$ are maximally distinguishable! In actual fact, if you instead compute the Kolmogorov distance, which *is* a good measure of distinguishability, you will find

$$d\left(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|\right) = \frac{1}{2}\left\||\psi\rangle\langle\psi| - |\tilde{\psi}\rangle\langle\tilde{\psi}|\right\|_{tr} = \frac{1}{2}\||\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|\|_{tr} = 0,$$

which indeed tells us that $|\psi\rangle$ and $|\tilde{\psi}\rangle$ are indistinguishable.

1.2 The power of noncontextuality

Quantum mechanics tells us that, if we have commuting operators, we can simultaneously measure them. In fact, we can simultaneously measure any function of the commuting operators, for example, the product of all of them. It is hence reasonable to expect that, if we have a deterministic assignment of measurement results of commuting operators (eg. a particular row or column of the nine two-qubit observables), the product of these measurement results will be equal to the result of measuring the product of the operators. However, in this question, we will show that this is incompatible with the noncontextual nature of quantum mechanics.

(a) Using the facts $I^2 = X^2 = Y^2 = Z^2 = I$, $XYZ = (iZ)Z = iI$ and $YXZ = (-iZ)Z = -iI$, we can easily compute the products of the individual rows and columns:

	Observables			Products of rows
	$X \otimes I$	$I \otimes X$	$X \otimes X$	$I \otimes I$
	$I \otimes Y$	$Y \otimes I$	$Y \otimes Y$	$I \otimes I$
	$X \otimes Y$	$Y \otimes X$	$Z \otimes Z$	$XYZ \otimes YXZ = I \otimes I$
Products of columns	$I \otimes I$	$I \otimes I$	$XYZ \otimes XYZ = -I \otimes I$	

Notice that, apart from the last column, all the products are $I \otimes I$, which only has +1 as its eigenvalue, and hence corresponds to an even number of -1's in the corresponding row or column. The product of the last column is $-I \otimes I$ which only has -1 as its eigenvalue and hence corresponds to an odd number of -1's in the last column.

(b) Suppose there is a deterministic assignment $\alpha_{ij} = +1$ or -1:

$$\begin{array}{lll} \alpha_{11} = \langle X \otimes I \rangle & \alpha_{12} = \langle I \otimes X \rangle & \alpha_{13} = \langle X \otimes X \rangle \\ \alpha_{21} = \langle I \otimes Y \rangle & \alpha_{22} = \langle Y \otimes I \rangle & \alpha_{23} = \langle Y \otimes Y \rangle \\ \alpha_{31} = \langle X \otimes Y \rangle & \alpha_{32} = \langle Y \otimes X \rangle & \alpha_{33} = \langle Z \otimes Z \rangle, \end{array}$$

For this assignment to also be noncontextual, each value of α_{ij} must simultaneously satisfy the constraints from part (a) from both the row and the column it is in, so that it does not matter whether if we choose to complete the set of orthogonal projectors with observables from the row or from the column. This means that the set $\{\alpha_{ij}\}$ must satisfy:

$$\begin{aligned} \text{rows:} \quad & \alpha_{11}\alpha_{12}\alpha_{13} = 1 && \text{(S1)} \\ & \alpha_{21}\alpha_{22}\alpha_{23} = 1 && \text{(S2)} \\ & \alpha_{31}\alpha_{32}\alpha_{33} = 1 && \text{(S3)} \\ \text{columns:} \quad & \alpha_{11}\alpha_{21}\alpha_{31} = 1 && \text{(S4)} \\ & \alpha_{12}\alpha_{22}\alpha_{32} = 1 && \text{(S5)} \\ & \alpha_{13}\alpha_{23}\alpha_{33} = -1 && \text{(S6)} \end{aligned}$$

Notice that

$$\begin{aligned} \text{(S1)} \times \text{(S2)} \times \text{(S3)}: \quad & \alpha_{11}\alpha_{12}\alpha_{13}\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{31}\alpha_{32}\alpha_{33} = +1, \\ \text{(S4)} \times \text{(S5)} \times \text{(S6)}: \quad & \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{12}\alpha_{22}\alpha_{32}\alpha_{13}\alpha_{23}\alpha_{33} = -1, \end{aligned}$$

which are two inconsistent equations, since the LHS's of both equations are identical, but the RHS's are not. Therefore, there cannot exist a deterministic assignment that is also noncontextual.

1.3 Schmidt-decomposable states.

(a) Given the state $|\psi\rangle_{AB}$, we can write it in Schmidt form in some ON bases, $\{|\mu\rangle_A\}_{\mu=1}^N$ and $\{|\mu\rangle_B\}_{\mu=1}^N$ respectively for $\mathcal{H}_A$ and $\mathcal{H}_B$:

$$|\psi\rangle_{AB} = \sum_{\mu=1}^{N} \sqrt{\lambda_\mu}\, |\mu\rangle_A \otimes |\mu\rangle_B \tag{S7}$$

where $\lambda_\mu \geq 0\ \forall \mu$. Consider the following operators:

$$\begin{aligned} U_A &:= \sum_{i,\mu=1}^{N} \delta_{i\mu} |i\rangle_A \langle\mu| \qquad \text{operator on } \mathcal{H}_A, \\ U_B &:= \sum_{i,\mu=1}^{N} \delta_{i\mu} |i\rangle_B \langle\mu| \qquad \text{operator on } \mathcal{H}_B. \end{aligned}$$

These operators can be viewed as change of basis transformations $\{|\mu\rangle\} \longrightarrow \{|i\rangle\}$ and are thus unitaries. We can check this explicitly:

$$\begin{aligned} U_A^\dagger U_A &= \sum_{i\mu j\nu=1}^{N} \left(\delta_{j\nu}|\nu\rangle_A\langle j|\right)\left(\delta_{i\mu}|i\rangle_A\langle\mu|\right) \\ &= \sum_{i\mu j\nu=1}^{N} \delta_{i\mu}\delta_{j\nu}\delta_{ij}|\nu\rangle_A\langle\mu| \\ &= \sum_{\mu=1}^{N} |\mu\rangle_A\langle\mu| = I_A, \end{aligned}$$

similarly for U_B. Now, apply the local unitaries U_A and U_B to the state $|\psi\rangle_{AB}$:

$$\begin{aligned} U_A \otimes U_B|\psi\rangle_{AB} &= \sum_{i,\mu j,\nu=1}^{N} \delta_{i\mu}\delta_{j\nu}|i\rangle_A\langle\mu| \otimes |j\rangle_B\langle\nu| \left(\sum_{\eta=1}^{N} \sqrt{\lambda\eta}|\eta\rangle_A \otimes |\eta\rangle_B\right) \\ &= \sum_{i\mu j\nu\eta} \delta_{i\mu}\delta_{j\nu}\delta_{\mu\eta}\delta_{\nu\eta}\sqrt{\lambda_\eta}|i\rangle_A \otimes |j\rangle_B \\ &= \sum_{i=1}^{N} \sqrt{\lambda_i}|i\rangle_A \otimes |i\rangle_A \\ &=: |\psi'\rangle_{AB} \qquad \text{(desired form in eq. (10))}. \end{aligned}$$

(b) From part (a), we see that we can transform from the given state $|\psi\rangle_{AB}$ to any other pure state on the bipartite Hilbert space with the *same* Schmidt coefficients as $|\psi\rangle_{AB}$, using only local unitaries. In other words, any bipartite pure state with the same Schmidt coefficients as $|\psi\rangle_{AB}$ is in its orbit under the action of local unitaries. We want to make sure that this result makes sense in terms of parameter counting, i.e. $U_A \otimes U_B|\psi\rangle_{AB}$ should have the right number of parameters required to specify any vector (pure state) in the Hilbert space with the same Schmidt coefficients as $|\psi\rangle_{AB}$.

To do this, let us first figure out how many parameters are needed to specify a unitary matrix. Any $N \times N$ unitary matrix U can be written as

$$U = e^{iQ} := \sum_{n=0}^{\infty} \frac{(iQ)^n}{n!},$$

where Q is a self-adjoint matrix. This means that we only need to specify a self-adjoint Q to specify U. So, how many real parameters are needed to specify Q? Since Q is self-adjoint, its diagonal entries are real, and there are N of them. Off-diagonal entries are specified by $N(N-1)/2$ complex parameters, i.e. $N(N-1)$ real parameters. So, altogether, we

need $N+N(N-1) = N^2$ real parameters to specify Q and hence U. From this, we see that $U_A \otimes U_B$ is parameterized by $2N^2$ real numbers.

However, this does *not* mean that $|\psi\rangle_{AB}$ has an orbit dimension $2N^2$, because there may be many different $U_A \otimes U_B$'s that act in the same way on a given $|\psi\rangle_{AB}$. Suppose we have two such local unitaries. How are they related to each other? Consider

$$U'_A \otimes U'_B |\psi\rangle_{AB} = U_A \otimes U_B |\psi\rangle_{AB},$$

then, multiplying $(U_A \otimes U_B)^{-1} = U_A^\dagger \otimes U_B^\dagger$ on both sides,

$$(U_A^\dagger \otimes U_B^\dagger)(U'_A \otimes U'_B)|\psi\rangle_{AB} = |\psi\rangle_{AB}.$$

We see that $(U_A^\dagger \otimes U_B^\dagger)(U'_A \otimes U'_B) = U_A^\dagger U'_A \otimes U_B^\dagger U'_B$ acts trivially (i.e. as the identity) on $|\psi\rangle_{AB}$. We call such an operator a *stabilizer* of $|\psi\rangle_{AB}$, and such operators form a subgroup of the group of local unitaries, called the stabilizer group of $|\psi\rangle_{AB}$. Two local unitaries differing (on the right) only by an element in the stabilizer group will act in the same way on $|\psi\rangle_{AB}$, so we need to "mod out" this degeneracy. In group-theoretic terms, the stabilizer group partitions the group of local unitaries into equivalence classes according to the action on $|\psi\rangle_{AB}$. So, to determine the orbit dimension of $|\psi\rangle_{AB}$ under local unitaries, we need to first determine the dimension of its stabilizer group.

To do this, let us look at local unitaries differing infinitesimally from the identity:

$$U = e^{i\epsilon Q}$$

where ϵ is an infinitesimal (real) parameter, and Q is self-adjoint as before. Operators of this form are the generators of the unitary group, and finite unitaries can be obtained from these by exponentiation. We can Taylor expand U, keeping terms up to linear order in ϵ:

$$U = I + i\epsilon Q.$$

Writing $U_A \otimes U_B$ in this form:

$$\begin{aligned} U_A \otimes U_B &= (I_A + i\epsilon_A Q_A) \otimes (I_B + i\epsilon_B Q_B) \\ &= I_A \otimes I_B + i(\epsilon_A Q_A \otimes I_B + \epsilon_B I_A \otimes Q_B). \end{aligned} \tag{S8}$$

Now, $U_A \otimes U_B$ is in the stabilizer group of $|\psi\rangle_{AB}$ iff $(\epsilon_A Q_A \otimes I_B + \epsilon_B I_A \otimes Q_B)$ annihilates $|\psi\rangle_{AB}$, so that we are just left with $I_A \otimes I_B$ in eq. (S8). Therefore, we need $\epsilon_A Q_A$ and $\epsilon_B Q_B$ to satisfy (using the Schmidt form of $|\psi\rangle_{AB}$ from eq. (S7)):

$$\begin{aligned} & (\epsilon_A Q_A \otimes I_B + \epsilon_B I_A \otimes Q_B)|\psi\rangle_{AB} = 0 \\ \Rightarrow \quad & \sum_{\mu=1}^{N} \sqrt{\lambda_\mu}\, (\epsilon_A Q_A |\mu\rangle_A \otimes |\mu\rangle_B + \epsilon_B |\mu\rangle_A \otimes Q_B |\mu\rangle_B) = 0. \end{aligned} \tag{S9}$$

Taking inner product of this equation with ${}_A\langle\nu| \otimes {}_B\langle\tau|$:

$$\begin{aligned} 0 &= \sum_{\mu=1}^{N} \sqrt{\lambda_\mu}\,({}_A\langle\nu|\epsilon_A Q_A|\mu\rangle_A \delta_{\mu\tau} + {}_B\langle\tau|\epsilon_B Q_B|\mu\rangle_B \delta_{\mu\nu}) \\ &= \sqrt{\lambda_\tau}\,\epsilon_A \langle\nu|Q_A|\tau\rangle + \sqrt{\lambda_\nu}\,\epsilon_B \langle\tau|Q_B|\nu\rangle \end{aligned}$$

$$\Rightarrow \qquad \sqrt{\lambda_\tau}\,\epsilon_A \langle\nu|Q_A|\tau\rangle = -\sqrt{\lambda_\nu}\,\epsilon_B \langle\tau|Q_B|\nu\rangle. \tag{S10}$$

For a generic vector in Schmidt form, $\lambda_\nu \neq \lambda_\tau$ for $\nu \neq \tau$. This implies

$$\langle\nu|Q_A|\tau\rangle = 0 = \langle\tau|Q_B|\nu\rangle \qquad \text{for } \nu \neq \tau,$$

which tells us that Q_A and Q_B are diagonal matrices. Furthermore, eq. (S10) imposes a relation between Q_A and Q_B:

$$\begin{aligned} & \langle\nu|\epsilon_A Q_A|\nu\rangle = -\langle\nu|\epsilon_B Q_B|\nu\rangle \qquad \forall\nu \\ \Rightarrow \qquad & \epsilon_A Q_A = -\epsilon_B Q_B. \end{aligned} \tag{S11}$$

Since there are exactly N linearly independent diagonal self-adjoint matrices, there are exactly N linearly independent solutions to eq. (S11). This means that the stabilizer group of $|\psi\rangle_{AB}$ has dimension N.

Finally, we can put all these together and figure out the orbit dimension of $|\psi\rangle_{AB}$. We know we need $2N^2$ real parameters to specify $U_A \otimes U_B$, but we need to subtract away the dimension of the stabilizer group, which leaves us with $2N^2 - N$ real parameters.

How does this compare with the number of parameters needed to specify a vector with the same Schmidt coefficients as $|\psi\rangle_{AB}$? $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$ has *complex* dimension N^2, so any vector $|\varphi\rangle \in \mathcal{H}$ is specified by $2N^2$ *real* parameters. However, for a vector with the same Schmidt coefficients as $|\psi\rangle_{AB}$, we only have the freedom to specify its Schmidt basis, but not its Schmidt coefficients (these are real and there are N of them), so it can be described by $2N^2 - N$ real parameters. This is exactly the same number we got above.

(c) Let us denote a generic tripartite Schmidt-decomposable vector by

$$|\psi\rangle_{ABC} = \sum_{i=1}^{N} \sqrt{\lambda_i}|i\rangle_A \otimes |i\rangle_B \otimes |i\rangle_C.$$

We follow the same method as in part (b). Any tripartite local unitaries $U_A \otimes U_B \otimes U_C$ can be specified by $3N^2$ real parameters. To find the dimension of the stabilizer group, we again look at infinitesimal unitaries:

$$\begin{aligned} U_A \otimes U_B \otimes U_C = I_A \otimes I_B \otimes I_C + i\,(&\epsilon_A Q_A \otimes I_B \otimes I_C \\ &+ \epsilon_B I_A \otimes Q_B \otimes I_C + \epsilon_C I_C \otimes I_B \otimes Q_C)\,. \end{aligned}$$

$U_A \otimes U_B \otimes U_C$ is in the stabilizer iff

$$(\epsilon_A Q_A \otimes I_B \otimes I_C + \epsilon_B I_A \otimes Q_B \otimes I_C + \epsilon_C I_C \otimes I_B \otimes Q_C)|\psi\rangle_{ABC} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{N} \sqrt{\lambda_i}\,(\epsilon_A Q_A|i\rangle_A \otimes |i\rangle_B \otimes |i\rangle_C + \epsilon_B |i\rangle_A \otimes Q_B|i\rangle_B \otimes |i\rangle_C + \epsilon_C |i\rangle_A \otimes |i\rangle_B \otimes Q_C|i\rangle_C) = 0.$$

Taking inner product with ${}_A\langle j| \otimes {}_B\langle k| \otimes {}_C\langle l|$, we have:

$$\sqrt{\lambda_k}\delta_{kl}\langle j|\epsilon_A Q_A|k\rangle + \sqrt{\lambda_l}\delta_{jl}\langle k|\epsilon_B Q_B|l\rangle + \sqrt{\lambda_j}\delta_{jk}\langle l|\epsilon_C Q_C|j\rangle = 0. \quad \text{(S12)}$$

If $j \neq k$, eq. (S12) becomes

$$\sqrt{\lambda_k}\delta_{kl}\langle j|\epsilon_A Q_A|k\rangle + \sqrt{\lambda_l}\delta_{jl}\langle k|\epsilon_B Q_B|l\rangle = 0.$$

Furthermore, if $k = l$ (this means $j \neq l$ since $j \neq k$), we have

$$\sqrt{\lambda_k}\langle j|\epsilon_A Q_A|k\rangle = 0 \qquad \forall j \neq k$$

which tells us that Q_A is diagonal. A similar argument shows that Q_B and Q_C are also diagonal. So, Q_A, Q_B and Q_C are all diagonal, self-adjoint matrices, each of which can be described by N real parameters, so we need $3N$ real parameters altogether. However, they must also satisfy eq. (S12), which represents N constraints, leaving us with $3N - N = 2N$ as the dimension of the stabilizer group of $|\psi\rangle_{ABC}$.

This immediately tells us that the orbit of $|\psi\rangle_{ABC}$ under local unitaries is described by $3N^2 - 2N$ real parameters. For qubits, $N = 2$, so the dimension of the orbit is $3(2^2) - 2(2) = 8$.

(d) A tripartite Schmidt vector written in the form of eq. (12) has N (real) Schmidt coefficients, and from part (c), any vector in its orbit is further specified by $3N^2 - 2N$ real parameters. This means that the (real) dimension of the space of Schmidt-decomposable vectors is $3N^2 - 2N + N = 3N^2 - N$. For qubits ($N = 2$), this is equal to 10 real dimensions. However, the real dimension of the three-qubit Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C = \mathbb{C}^8$ is $2N^3 = 16$, so the codimension of the space of Schmidt-decomposable states is $16 - 10 = 6$. This means that, given a generic Schmidt-decomposable three-qubit state, there is a 6-dimensional space of states that we can never reach by local unitaries!

1.4 Completeness of subsystem correlations

(a) From Problem 1.3(b), we know that the space of self-adjoint matrices acting on a Hilbert space of dimension N is N^2-dimensional. Therefore, since $\{M_a\}_{a=1}^{N^2}$ is a set of N^2 linearly independent self-adjoint operators,

it forms a basis for self-adjoint operators on $\mathcal{H}$. In particular, for any $|\varphi\rangle \in \mathcal{H}$, we can expand $|\varphi\rangle\langle\varphi|$ in terms of $\{M_a\}$ as

$$|\varphi\rangle\langle\varphi| = \sum_{a=1}^{N^2} \lambda_a M_a, \qquad \lambda_a \in \mathbb{R}.$$

Then,

$$\langle\varphi|\rho|\varphi\rangle = \mathrm{tr}\left(\rho|\varphi\rangle\langle\varphi|\right) = \mathrm{tr}\left(\rho \sum_{a=1}^{N^2} \lambda_a M_a\right) = \sum_{a=1}^{N^2} \lambda_a \mathrm{tr}(\rho M_a).$$

Since $\mathrm{tr}(\rho M_a)$ is know $\forall a$, we can compute $\langle\varphi|\rho|\varphi\rangle$ using the above formula, for any $|\varphi\rangle \in \mathcal{H}$.

(b) There are a few ways to do this problem. I will give two different approaches and you can decide which you understand better. The first approach is very simple if you know a bit about bases of self-adjoint matrices. The second one is more computational.

Method 1:
Let me motivate this approach by first looking at the qubit case. From lecture, we know that any qubit state can be written as:

$$\rho = \frac{1}{2}\left(I + \vec{P}\cdot\vec{\sigma}\right)$$

where $\vec{P} := (P_x, P_y, P_z)$ is the (real) Bloch vector, and $\vec{\sigma} := (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Recall that the Pauli matrices satisfy:

$$\mathrm{tr}(\sigma_i^\dagger I) = \mathrm{tr}(\sigma_i) = 0$$
$$\text{and} \qquad \mathrm{tr}(\sigma_i^\dagger \sigma_j) = 2\delta_{ij} \qquad \forall i,j = x,y,z.$$

Using these two properties, we see that

$$P_i = \mathrm{tr}(\sigma_i^\dagger \rho), \qquad i = x,y,z. \tag{S13}$$

Furthermore, we know that we can write the Pauli matrices in terms of their eigenvectors (spectral decomposition):

$$\sigma_x = |+\rangle\langle+| - |-\rangle\langle-|$$
$$\sigma_y = |L\rangle\langle L| - |R\rangle\langle R|$$
$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

where $\{|0\rangle, |1\rangle\}$ is the standard qubit basis, $|\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|L,R\rangle := (|0\rangle \pm i|1\rangle)/\sqrt{2}$. Then, eq. (S13) can then be re-written as:

$$P_x = \mathrm{tr}(\sigma_x^\dagger \rho) = \langle+|\rho|+\rangle - \langle-|\rho|-\rangle$$
$$P_y = \mathrm{tr}(\sigma_y^\dagger \rho) = \langle L|\rho|L\rangle - \langle R|\rho|R\rangle$$
$$P_z = \mathrm{tr}(\sigma_z^\dagger \rho) = \langle 0|\rho|0\rangle - \langle 1|\rho|1\rangle.$$

Notice that the RHS's of all three equations are in terms of diagonal matrix elements of ρ which we know.

We can do something similar for the general N-dimensional case, using the basis $\{M_a\}_{a=1}^{N^2}$ for self-adjoint operators from part (a) in place of $\{I, \sigma_x, \sigma_y, \sigma_z\}$ for qubits. In addition, we can choose to use a basis that is orthonormal in the following sense:

$$\mathrm{tr}(M_a^\dagger M_b) = \delta_{ab} C \tag{S14}$$

for some overall normalization constant C. Since any state ρ is self-adjoint, we can expand it in this basis:

$$\rho = \sum_{a=1}^{N^2} \alpha_a M_a, \qquad \alpha_a \in \mathbb{R}.$$

Using the orthonormality condtion (S14), we see that

$$\alpha_a = \mathrm{tr}(M_a^\dagger \rho), \qquad \forall a \in \{1, \ldots, N^2\}. \tag{S15}$$

Since M_a is self-adjoint, it is diagonalizable and has spectral decomposition:

$$M_a = \sum_{i=1}^{N} m_{ai} |i_a\rangle\langle i_a|$$

where $\{|i_a\rangle\}_{i=1}^N$ is an ON basis for $\mathcal{H}$. Using this in eq. (S15), we can write:

$$\alpha_a = \sum_{i=1}^{N} m_{ai} \langle i_a|\rho|i_a\rangle.$$

The RHS is completely in terms of diagonal matrix elements of ρ which we know, and hence we know ρ.

Method 2: (Solution by Andrew Landahl (1998))
To know ρ, we must not only know its diagonal elements in some basis $\{|j\rangle\}$, we also need to know the real and imaginary components of its off-diagonal matrix elements in the same basis. Fortunately, we can find these off-diagonal elements in terms of diagonal ones. For any j, k, consider the states:

$$|\psi_{jk}\rangle := \frac{1}{\sqrt{2}} \left(|j\rangle + |k\rangle\right),$$
$$|\tau_{jk}\rangle := \frac{1}{\sqrt{2}} \left(|j\rangle + i|k\rangle\right).$$

Looking at $\langle\psi_{jk}|\rho|\psi_{jk}\rangle$, we find that we can express it as

$$\begin{aligned}\langle\psi_{jk}|\rho|\psi_{jk}\rangle &= \frac{1}{2}\left(\langle j| + \langle k|\right)\rho\left(|j\rangle + |k\rangle\right)\\ &= \frac{1}{2}\left[\langle j|\rho|j\rangle + \langle k|\rho|k\rangle + \left(\langle j|\rho|k\rangle + \langle k|\rho|j\rangle\right)\right]\\ &= \frac{1}{2}\left[\langle j|\rho|j\rangle + \langle k|\rho|k\rangle + \left(\langle j|\rho|k\rangle + \langle j|\rho|k\rangle^*\right)\right]\\ &= \frac{1}{2}\left(\langle j|\rho|j\rangle + \langle k|\rho|k\rangle\right) + \mathrm{Re}\{\langle j|\rho|k\rangle\}.\end{aligned}$$

Similarly, we can write $\langle\tau_{jk}|\rho|\tau_{jk}\rangle$ as:

$$\langle\tau_{jk}|\rho|\tau_{jk}\rangle = \frac{1}{2}\left(\langle j|\rho|j\rangle + \langle k|\rho|k\rangle\right) - \mathrm{Im}\{\langle j|\rho|k\rangle\}.$$

These relations allow us to express the off-diagonal elements of ρ as:

$$\begin{aligned}\langle j|\rho|k\rangle &= \mathrm{Re}\{\langle j|\rho|k\rangle\} + i\mathrm{Im}\{\langle j|\rho|k\rangle\}\\ &= \langle\psi_{jk}|\rho|\psi_{jk}\rangle - i\langle\tau_{jk}|\rho|\tau_{jk}\rangle - \frac{1}{2}(1-i)\left(\langle j|\rho|j\rangle + \langle k|\rho|k\rangle\right).\end{aligned}$$

The RHS consists only of known quantities of the form $\langle\varphi|\rho|\varphi\rangle$.

(c) We can do this problem by dimension counting. From the analysis in Problem 1.3(b), we know that an $N\times N$ self-adjoint matrix is specified by N^2 real parameters, so there are exactly N^2 linearly independent elements in a basis for self-adjoint matrices. Let the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$ be d_A and d_B respectively, so that $\dim(\mathcal{H}_A\otimes\mathcal{H}_B) = d_Ad_B$. Then, self-adjoint operators on $\mathcal{H}_A\otimes\mathcal{H}_B$ are $d_Ad_B\times d_Ad_B$ matrices, and hence, a basis for them will have $d_A^2d_B^2$ elements. By the same logic, $\{M_a\}$ has d_A^2 elements, and $\{N_b\}$ has d_B^2 elements, so there are $d_A^2d_B^2$ linearly independent elements in $\{M_a\otimes N_b\}$, which is exactly the right number of elements in a basis for self-adjoint operators on $\mathcal{H}_A\otimes\mathcal{H}_B$. Orthogonality of $\{M_a\otimes N_b\}$ follows directly from orthogonality of $\{M_a\}$ and $\{N_b\}$.

(d) **Claim:** If $\{M_a^i\}$ is a basis for self-adjoint operators on $\mathcal{H}_i$, then $\{\bigotimes_{i=1}^n M_a^i\}$ is a basis for self-adjoint operators on $\{\bigotimes_{i=1}^n H_i\}$.

Proof: We prove this by induction, with part (c) as the base case for $n=2$. Suppose the claim is true for some $n=k$. Let d_k be the dimension of $\{\bigotimes_{i=1}^k H_i\}$, and let p be the dimension of $\mathcal{H}_{k+1}$. The basis of self-adjoint operators on $\{\bigotimes_{i=1}^k H_i\}$ is $\{\bigotimes_{i=1}^k M_a^i\}$, and it has d_k^2 elements, whereas the basis for $\mathcal{H}_{k+1}$ is $\{M_a^{k+1}\}$, with p^2 elements. The combined Hilbert space $\{\bigotimes_{i=1}^{k+1} H_i\}$ has dimension d_kp, so its self-adjoint basis has $d_k^2p^2$ elements. On the other hand, there are $d_k^2p^2$ orthogonal elements in $\{\left(\bigotimes_{i=1}^k M_a^i\right)\bigotimes M_b^{k+1}\} = \{\bigotimes_{i=1}^{k+1} M_a^i\}$, which is exactly the right number for a basis of self-adjoint operators on $\{\bigotimes_{i=1}^{k+1} H_i\}$.

(e) For a real Hilbert space, the observables will have a spectral decomposition that contains only real coefficients, and are hence described by real, symmetric matrices instead of self-adjoint ones. A real, symmetric matrix $M^T = M$ on an N-dimensional real Hilbert space $\mathcal{H}$ has N diagonal entries, and $N(N-1)/2$ *real* off-diagonal entries. This means that a basis for real, symmetric matrices consists of $N+N(N-1)/2 = N(N+1)/2$ linearly independent elements. For $\mathcal{H}\otimes\mathcal{H}$, the basis for real symmetric matrices is $(N^2)(N^2+1)/2 = [N(N+1)/2]^2+[N(N-1)/2]^2 > [N(N+1)/2]^2$, strictly greater than the number of elements in $\{M_a \otimes M_b\}$, so (c) is no longer true. Therefore, having full information about the expectation values of observables of the form $\{M_a \otimes M_b\}$ is no longer equivalent to having full information about the state ρ_{AB}, i.e. subsystem correlations are no longer complete. What they miss are correlations coming from global observables of the form $A_a \otimes A_b$, where A_a and A_b are anti-symmetric operators (and hence unobservable locally). Notice that such an operator is symmetric on the combined space, and is hence a valid global observable.