Ph219a/CS219a

Solutions
Nov 9, 2008

Problem 1.1

Given that Alice and Bob share the pure state $|\psi\rangle$ which is Schmidt decomposed as

$$|\psi\rangle = \sum_i \sqrt{p_i}\, |\alpha_i\rangle \otimes |\beta_i\rangle \tag{1}$$

In terms of the Schmidt bases for $|\psi\rangle$, a projective measurement done by Bob (E_a^B) can be expressed as

$$E_a^B = \sum_{i,j} m_{ij}\, |\beta_i\rangle \langle\beta_j| \tag{2}$$

If Alice is to do a measurement such that outcome a occurs with the same probability as in Bob's measurement, the two projectors must have the same expectation value in state $|\psi\rangle$ -

$$\begin{aligned} \langle\psi| E_a^A |\psi\rangle &= \langle\psi| E_a^B |\psi\rangle \\ \Rightarrow \sum_i \langle\alpha_i| E_a^A |\alpha_i\rangle &= \sum_i \langle\beta_i| E_a^B |\beta_i\rangle \end{aligned} \tag{3}$$

This is easily achieved if Alice chooses her projective measurement E_a^A identical to Bob's so that

$$E_a^A = \sum_{i,j} m_{ij}\, |\alpha_i\rangle \langle\alpha_j| \tag{4}$$

Two states are *locally equivalent* if they have the same Schmidt coefficients. The unnormalized state after Bob's projective measurement, expressed in the Schmidt form is

$$|\psi_a\rangle = (\mathcal{I} \otimes E_a^B)\, |\psi\rangle = \sum_{i,k} \sqrt{p_k} m_{ik}\, |\alpha_k\rangle \otimes |\beta_i\rangle \tag{5}$$

Whereas Alice's measurement gives the state

$$|\psi'_a\rangle = (E_a^A \otimes \mathcal{I})\, |\psi\rangle = \sum_{i,k} \sqrt{p_k} m_{ik}\, |\alpha_i\rangle \otimes |\beta_k\rangle \tag{6}$$

Notice that these two states are the same up to a SWAP ie. $|\alpha_i\rangle \otimes |\beta_k\rangle \to |\alpha_k\rangle \otimes |\beta_i\rangle$. Thus, if $U_a^A \otimes U_a^B$ rotates $|\psi_a\rangle$ to its Schmidt form, then $U_a^B \otimes U_a^A$ will take $|\psi'_a\rangle$ into its Schmidt form. Therefore, the two final states are locally equivalent if

$$(U_a^A \otimes U_a^B)(\mathcal{I} \otimes E_a^B)\, |\psi\rangle = (U_a^B \otimes U_a^A)(E_a^A \otimes \mathcal{I})\, |\psi\rangle \tag{7}$$

(Here's a more detailed argument to justify this claim - The unitaries that rotate $|\psi_a\rangle$ into its Schmidt form are obtained by a singular value decomposition of the matrix of coefficients, ie. $\{C_{ik} = \sqrt{p_k} m_{ik}\}$. Since $|\psi'_a\rangle$ has the same coefficients as $|\psi_a\rangle$ in the $\{|\alpha_i\rangle, |\beta_i\rangle\}$ basis , the unitaries that transform into its Schmidt form are the same, except for a swapping of subsystems to account for the swap of the basis state indices.)
Equation (7) implies

$$(\mathcal{I} \otimes E_a^B) |\psi\rangle = ((U_a^A)^{-1} U_a^B \otimes (U_a^B)^{-1} U_a^A) |\psi\rangle \tag{8}$$

So the unitaries that Alice has to perform are given by $V_a^A = (U_a^A)^{-1} U_a^B$ and $V_a^B = (U_a^B)^{-1} U_a^A$.

Problem 1.2

(i) First we prove the converse -
From the GJHW theorem we know that two different realizations of a density operator ρ -

$$\begin{aligned} \rho &= \sum_i p_i |\alpha_i\rangle \langle\alpha_i| \\ &= \sum_\mu q_\mu |\phi_\mu\rangle \langle\phi_\mu| \end{aligned} \tag{9}$$

are related by

$$\sqrt{q_\mu} |\phi_\mu\rangle = \sum_i \sqrt{p_i} V_{\mu i} |\alpha_i\rangle \tag{10}$$

for some unitary V. Computing the norm on both sides of equation (10), we have

$$q_\mu = \sum_i p_i |V_{\mu i}|^2 = \sum_i p_i D_{\mu i} \tag{11}$$

The matrix D with elements $D_{\mu i} = |V_{\mu i}|^2$ is doubly stochastic (since V is unitary), thus proving that $q \prec p$.

(ii) The forward direction -
Given probability vectors q and p such that $q \prec p$, we know from Horn's lemma that there exists a unitary V such that

$$q_\mu = \sum_i p_i |V_{\mu i}|^2 \tag{12}$$

Consider the density operator ρ expressed in its diagonal form as $\rho = \sum_i p_i |\alpha_i\rangle \langle\alpha_i|$. Using the unitary V, we can construct pure states $|\phi_\mu\rangle$ as follows

$$\sqrt{q_\mu} |\phi_\mu\rangle = \sum_i \sqrt{p_i} V_{\mu i} |\alpha_i\rangle \tag{13}$$

Computing the norm on both sides, we find that the states $\{|\phi_\mu\rangle\}$ have unit norm -

$$\begin{aligned} q_\mu \langle\phi_\mu\rangle \phi_\mu &= \sum_i p_i |V_{\mu i}|^2 \\ \Rightarrow \langle\phi_\mu|\phi_\mu\rangle &= 1 \quad \text{(using equation (12))} \end{aligned} \tag{14}$$

Further,

$$\begin{aligned}\sum_\mu q_\mu |\phi_\mu\rangle\langle\phi_\mu| &= \sum_\mu \sum_{i,j} \sqrt{p_i p_j} V_{\mu i} V_{\mu j}^* |\alpha_i\rangle\langle\alpha_j| \\ &= \sum_i p_i |\alpha_i\rangle |\alpha_i\rangle \qquad (15)\end{aligned}$$

since $\sum_\mu V_{\mu i} V_{\mu j}^* = \delta_{ij}$ when V is unitary. Thus we have shown that $\exists$ pure, normalized $\{|\phi_\mu\rangle\}$ such that $\rho = \sum_\mu q_\mu |\phi_\mu\rangle\langle\phi_\mu|$.

Problem 1.3

From Problem (1.1) we know that if the deterministic transformation $|\Psi\rangle \to |\Phi\rangle$ is possible using a 2-LOCC protocol, it can also be achieved by a 1-LOCC protocol, ie. there exists a POVM with elements $\{M_\mu\}$ that Alice can perform, and a superoperator ε_μ conditioned on Alice's outcome that Bob can perform, such that

$$(\mathcal{I} \otimes \varepsilon_\mu)(M_\mu |\Psi\rangle\langle\Psi| M_\mu^\dagger) \propto |\Phi\rangle\langle\Phi| \qquad (16)$$

where $\sum_\mu M_\mu M_\mu^\dagger = \mathcal{I}$. Note that Alice's local unitary operation is included in constructing the generalized measurement M_μ.

Bob's superoperator cannot change Alice's reduced density operator ρ^A, so the reduced density operators before and after Bob's local operation are related as

$$M_\mu \rho_\psi^A M_\mu^\dagger = c_\mu \rho_\Phi^A \qquad (17)$$

where $\rho_\Psi^A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$, $\rho_{Phi}^A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$ and $\sum_\mu c_\mu = 1$ (since c_μ is the probability of outcome μ.)

Now polar decomposing $M_\mu \sqrt{\rho_\Phi^A}$, we have a unitary U_μ such that

$$M_\mu \sqrt{\rho_\Psi^A} = \sqrt{M_\mu\, \rho_\Psi^A\, M_\mu^\dagger} U_\mu = \sqrt{c_\mu \rho_\Phi^A} U_\mu \qquad (18)$$

where we have used equation (17). Now consider the following

$$\begin{aligned}\rho_\Psi^A = \sqrt{\rho_\Psi^A} \mathcal{I} \sqrt{\rho_\Psi^A} &= \sqrt{\rho_\Psi^A}\,(\sum_\mu M_\mu^\dagger M_\mu)\,\sqrt{\rho_\Psi^A} \\ &= \sum_\mu (\sqrt{\rho_\Psi^A}\, M_\mu^\dagger)(M_\mu \sqrt{\rho_\Psi^A}) \\ &= \sum_\mu c_\mu U_\mu^\dagger \rho_\Phi^A U_\mu \qquad (19)\end{aligned}$$

using the polar decomposition described earlier. Thus we have shown that ρ_Ψ^A is obtained from ρ_Φ^A by applying a unitary U_μ conditioned on measurement outcome μ.

Expressing Alice's initial and final density operators as

$$\begin{aligned}\rho^A_\Psi &= \sum_i p_i \left|\alpha_i\right\rangle \left\langle\alpha_i\right| \\ \rho^A_\Phi &= \sum_j q_j \left|\alpha'_j\right\rangle \left\langle\alpha'_j\right|\end{aligned} \tag{20}$$

and using equation (19), we have,

$$\begin{aligned}\rho^A_\Psi &= \sum_i p_i \left|\alpha_i\right\rangle \left\langle\alpha_i\right| \\ &= \sum_{\mu,j} c_\mu q_j U^\dagger_\mu (\left|\alpha'_j\right\rangle \left\langle\alpha'_j\right|) U_\mu\end{aligned} \tag{21}$$

Now inserting complete sets of states $\mathcal{I} = \sum_i \left|\alpha_i\right\rangle \left\langle\alpha_i\right|$ and denoting $\left\langle\alpha'_j\right| U_\mu \left|\alpha_k\right\rangle = (U_\mu)_{jk}$, we have,

$$\begin{aligned}\sum_i p_i \left|\alpha_i\right\rangle \left\langle\alpha_i\right| &= \sum_{\mu,j,i,k} c_\mu q_j (U_\mu)^*_{ji} \left|\alpha_i\right\rangle \left\langle\alpha_k\right| (U_\mu)_{jk} \\ &= \sum_{\mu,j,i,k} c_\mu q_j (U_\mu)^*_{ji} (U_\mu)_{jk} \left|\alpha_i\right\rangle \left\langle\alpha_k\right| \\ \Rightarrow p_i &= \sum_{\mu,j} c_\mu q_j (U_\mu)^*_{ji} (U_\mu)_{ji} \\ &= \sum_j D_{ij} q_j\end{aligned} \tag{22}$$

where D with elements $D_{ij} = \sum_\mu c_\mu |(U_\mu)_{ji}|^2$ is a doubly stochastic matrix since the U_μ's are unitary and $\sum_\mu c_\mu = 1$

Problem 1.4

(a) Since $p_a = \mathrm{Tr}(\rho E_a)$ and $\tilde{p}_a = \mathrm{Tr}(\tilde{\rho} E_a)$,

$$d(p,\tilde{p}) = \frac{1}{2}\sum_{a=1}^{N} |p_a - \tilde{p}_a| = \frac{1}{2}|\mathrm{Tr}(\,(\rho - \tilde{\rho})E_a)| \tag{23}$$

Writing $\rho - \tilde{\rho}$ in its eigen-basis, we have $\rho - \tilde{\rho} = \sum_i \lambda_i \left|i\right\rangle \left\langle i\right|$, so that

$$\begin{aligned}d(p,\tilde{p}) &= \sum_{a=1}^{N} |\mathrm{Tr}(\,(\sum_i \lambda_i \left|i\right\rangle \left\langle i\right|) E_a)| \\ &= \frac{1}{2}\sum_{a=1}^{N} |\sum_i (\lambda_i \left\langle i\right| E_a \left|i\right\rangle)| \\ &\leq \frac{1}{2}\sum_i |\lambda_i| (\sum_{a=1}^{N} \left\langle i\right| E_a \left|i\right\rangle) \quad \text{(Triangle inequality!)} \\ &= \frac{1}{2}\sum_i |\lambda_i|\end{aligned} \tag{24}$$

using the fact that E_a are orthogonal projectors satisfying $\sum_a^N E_a = \mathcal{I}$.

(b) Construct one-dimensional projectors that project onto the basis in which $\rho - \tilde{\rho}$, ie. $E_i = |i\rangle\langle i|$. Notice that these are orthogonal projectors that sum to the identity, as desired. With this construction, the upper bound of equation (24) is saturated -

$$\begin{aligned} d(p,\tilde{p}) &= \sum_j |\sum_i (\lambda_i \langle i| E_j |i\rangle)| \\ &= \sum_j |\sum_i (\lambda_i \delta_{ij})| \\ &= \sum_j |\lambda_j| \end{aligned} \tag{25}$$

(c) Using the given definition, lets write down the L^1 norm of $\rho - \tilde{\rho}$:

$$\begin{aligned} \| \rho - \tilde{\rho} \|_1 &= \mathrm{Tr}(\sqrt{(\rho-\tilde{\rho})(\rho-\tilde{\rho})^\dagger}) \\ &= \mathrm{Tr}(\sqrt{(\rho-\tilde{\rho})^2}) \ \text{ (since } \rho - \tilde{\rho} \text{ is Hermitian)} \\ &= \sum_i |\lambda_i| \\ &= 2d(p,\tilde{p}) \end{aligned} \tag{26}$$

Thus we have,

$$d(p,\tilde{p}) = \frac{1}{2} \| \rho - \tilde{\rho} \|_1 \tag{27}$$

(d) Using the given phase convention,

$$\rho - \tilde{\rho} = |\psi\rangle\langle\psi| - |\tilde{\psi}\rangle\langle\tilde{\psi}| = \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix} \tag{28}$$

The eigenvalues are simply $\pm\cos\theta$ so that

$$d(p,\tilde{p}) = \frac{1}{2}(|\cos\theta| + |-\cos\theta|) = |\cos\theta| \tag{29}$$

(e) The Hilbert space norm of $(|\psi\rangle - |\tilde{\psi}\rangle)$ is given by

$$\begin{aligned} \| \, |\psi\rangle - |\tilde{\psi}\rangle \, \|^2 &= (\langle\psi| - \langle\tilde{\psi}|)(|\psi\rangle - |\tilde{\psi}\rangle) \\ &= \langle\psi|\psi\rangle - (\langle\tilde{\psi}|\psi\rangle + \langle\psi|\tilde{\psi}\rangle) + \langle\tilde{\psi}|\tilde{\psi}\rangle \\ &= 2(1-\sin\theta) \end{aligned} \tag{30}$$

Comparing this with the result of part (d),

$$\begin{aligned} (\, d(|\psi\rangle\langle\psi|\,, |\tilde{\psi}\rangle\langle\tilde{\psi}|)\,)^2 &= \cos^2\theta \\ &= 1 - \sin^2\theta = (1-\sin\theta)(1+\sin\theta) \\ &\leq 2(1-\sin\theta) \end{aligned} \tag{31}$$

implying that $d(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|) \leq \| \, |\psi\rangle - |\tilde{\psi}\rangle \, \|$.

(f) Consider for example, the states $|\psi\rangle$ and $|\tilde{\psi}\rangle$ for $\theta = \frac{3\pi}{2}$ so that $|\psi\rangle = -|\tilde{\psi}\rangle$. Since quantum states are really *rays*, vectors which differ only by an overall phase factor represent the same physical state. In other words, for $\theta = \frac{3\pi}{2}$, $|\psi\rangle$ and $|\tilde{\psi}\rangle$ should be indistinguishable. But computing the Hilbert space norm $\| \, |\psi\rangle - |\tilde{\psi}\rangle \, \|$, we find,

$$\| \, |\psi\rangle - |\tilde{\psi}\rangle \, \| = \| \, |\psi\rangle + |\psi\rangle \, \| = 2 \tag{32}$$

which would imply that $|\psi\rangle$ and $|\tilde{\psi}\rangle$ are maximally distinguishable! The norm $\| \, |\psi\rangle - |\tilde{\psi}\rangle \, \|$ is therefore not a good measure of distingushability.

On the other hand, distance measure d, also called the *Kolmogrov distance* between two states, gives:

$$d(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|) = \frac{1}{2} \, \| \, |\psi\rangle\langle\psi| - |\tilde{\psi}\rangle\langle\tilde{\psi}| \, \|_1 = \frac{1}{2} \, \| \, |\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| \, \|_1 = 0 \tag{33}$$

implying that the two states are indeed indistinguishable.