Ph219/CS219: Quantum Computation
Fall 2005
Solutions to Problem Set 3

Problem 3.1

(a) If there is a SLOCC protocol transforming $|\varphi\rangle \mapsto |\psi\rangle$, then there are local Kraus operators $\{M_\mu\}$ such that $|\psi\rangle\langle\psi| = \sum_\mu M_\mu|\varphi\rangle\langle\varphi|M_\mu^\dagger$ (up to a nonzero normalizing factor). The right-side is a sum of nonnegative operators and the left-hand side is rank 1, so all terms on the right-hand side are proportional to $|\psi\rangle\langle\psi|$. Choose one nonzero term. Any local Kraus operator can be expressed as tensor product $\tilde{A}\otimes\tilde{B}\otimes\tilde{C}$; therefore, after a rescaling, $|\psi\rangle = A\otimes B\otimes C|\varphi\rangle$. We only need to check that the matrices A, B, C are invertible.

Consider A. In the state $|\varphi\rangle$, denote the marginal density operator for the first qubit by ρ, and in the state $|\psi\rangle$, denote the marginal density operator for the first qubit by σ. Then $\sigma = A\rho A^\dagger$. Since σ has full rank, it has a trivial kernel; therefore so does $A^\dagger$. Thus $A^\dagger$ is invertible and so is A. The same argument also shows that B and C are invertible.

(b) The tensor product operator $A\otimes B\otimes C$ takes product vectors to product vectors. If invertible, it also preserves linear independence. So if $|\varphi\rangle$ is a linear combination of two independent product vectors, then so is $|\psi\rangle = A\otimes B\otimes C|\varphi\rangle$.

(c) In general, for a bipartite state $\sum_i |a_i\rangle\otimes|b_i\rangle$ (vectors not normalized), the marginal density operator ρ_B is

$$\rho_B = \sum_{i,j} |b_i\rangle\langle b_j|\langle a_j|a_i\rangle \ . \tag{1}$$

For any vector $|v\rangle$, clearly $\rho_B|v\rangle$ is a linear combination of the $\{|b_i\rangle\}$. Therefore, the rangle of ρ_B must be contained in the span of the $\{|b_i\rangle\}$.

Now in the case of a three-qubit pure state $|\varphi\rangle$, ρ_A and ρ_{BC} have the same rank, so that if ρ_A has full rank, then ρ_{BC} also has rank 2. If $|\varphi\rangle = |a_1\rangle\otimes|b_1\rangle\otimes|c_1\rangle + |a_2\rangle\otimes|b_2\rangle\otimes|c_2\rangle$, then the range of ρ_{BC} is two-dimensional and is contained in the (two-dimensional) span of the two product states $|b_1\rangle\otimes|c_1\rangle$ and $|b_2\rangle\otimes|c_2\rangle$; therefore, the range is equal to the span; it therefore contains both $|b_1\rangle\otimes|c_1\rangle$ and $|b_2\rangle\otimes|c_2\rangle$.

(d) For the GHZ state:

$$\rho_A = \rho_B = \rho_C = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \ , \tag{2}$$

and

$$\rho_{AB} = \rho_{BC} = \rho_{AC} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| \ . \tag{3}$$

For the W state:

$$\rho_A = \rho_B = \rho_C = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \ , \tag{4}$$

and

$$\rho_{AB} = \rho_{BC} = \rho_{AC} = \frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}|\psi^+\rangle\langle\psi^+| \ , \tag{5}$$

where $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$.

(e) For both the W state and the GHZ state, all the single-qubit marginal density operators have full rank. The GHZ state can be expressed as a combination of two linearly independent product states, $|000\rangle$ and $|111\rangle$. In the W state, the range of ρ_{BC} is spanned by $|00\rangle$ and $|\psi^+\rangle$, which are orthogonal, and therefore does not contain two linearly independent product states. Therefore, by (c), W state cannot be expressed as a combination of two linearly independent product states. By (b), no SLOCC protocol transforms GHZ $\mapsto$ W. A SLOCC protocol that transforms W $\mapsto$ GHZ would be SLOCC reversible. Therefore, no SLOCC protocol transforms W $\mapsto$ GHZ. The GHZ state and the W state are SLOCC inequivalent.

Problem 3.2

(a) To simulate $[q \to qq]$ using $[q \to q]$, Alice applies a CNOT with her input qubit as control and $|0\rangle$ as a target; hence $|x0\rangle_A \mapsto |xx\rangle_A$. Then Alice sends the second qubit to Bob ($|x\rangle_A \mapsto |x\rangle_B$).

To simulate $[c \to c]$, Alice throws her qubit away ($|x\rangle_A \mapsto |x\rangle_E$) after doing $[q \to qq]$.

To create an ebit ($[qq]$), Alice prepares an X eigenstate and does $[q \to qq]$ ($|0\rangle + |1\rangle \mapsto |00\rangle + |11\rangle$).

(b) Alice and Bob share the Bell pair $|\phi^+\rangle$. By applying a unitary to her input qubits $|xy\rangle_A$ and her half of the Bell pair, Alice transforms the state to $|xy\rangle_A \otimes (Z^x X^y \otimes I)|\phi^+\rangle$. Then Alice sends her half of the Bell pair to Bob, and Bob decodes the Bell pair with a CNOT and a Hadamard gate: $(Z^x X^y \otimes I)|\phi^+\rangle \mapsto |xy\rangle_B$. Now the state shared by Alice and Bob is $|xy\rangle_A \otimes |xy\rangle_B$; thus $2[q \to qq]$ has been achieved.

(c) The input state $|\psi\rangle_A$ can be sent from Alice to Bob by "coherent teleportation," where instead of Bell measurement, Alice applies a unitary that maps $|\phi^+\rangle \to |\phi^+\rangle \otimes |00\rangle$, $|\psi^+\rangle \to |\psi^+\rangle \otimes |01\rangle$, $|\phi^-\rangle \to |\phi^-\rangle \otimes |10\rangle$, $|\psi^-\rangle \to |\psi^-\rangle \otimes |11\rangle$. After $[q \to qq]$, Alice and Bob share

$$\frac{1}{2}\sum_{x,y}|xy\rangle_A\otimes|xy\rangle_B\otimes Z^xX^y|\psi\rangle_B\ . \tag{6}$$

After Bob applies a conditional Z controlled by x and a conditional X controlled by y, the state is $\frac{1}{2}\sum_{x,y}|xy\rangle_A\otimes|xy\rangle_B\otimes|\psi\rangle_B=|\phi^+\rangle_{AB}^{\otimes 2}\otimes|\psi\rangle_B$. Thus $[q\to q]$ has been achieved, and Alice and Bob now share $2[qq]$.

Problem 3.3

(a) The "if" part is obvious. For the "only if" part: If ρ_{AB} is separable, then $\rho_{AB}=\sum_{ij}p_{ij}\rho_{A,i}\otimes\rho_{B,j}$. Each of the density operators $\rho_{A,i}$, $\rho_{B,j}$ can be realized as an ensemble of pure states:

$$\rho_{AB}=\sum_{i,j,a,b}p_{ij}\ p_a^{(i)}\ p_b^{(j)}\rho_{A,ia}\otimes\rho_{B,jb}\ , \tag{7}$$

where each $\rho_{A,ia}$ and $\rho_{B,jb}$ is pure. This can be rewritten in the form

$$\rho_{AB}=\sum_\mu q_\mu\ \rho_{A,\mu}\otimes\rho_{B,\mu} \tag{8}$$

with the understanding that $\mu=(i,j,a,b)$, $q_\mu=p_{ij}\ p_a^{(i)}\ p_b^{(j)}$, $\rho_{A,\mu}=\rho_{A,ia}$, and $\rho_{B,\mu}=\rho_{B,jb}$.

(b) If ρ is a density operator, then so is its transpose ρ^T. (Both have the same trace and the same eigenvalues, though the eigenvectors are different if ρ is not real.) If ρ is separable, then

$$I\otimes T:\rho_{AB}=\sum_{ij}p_{ij}\ \rho_{A,i}\otimes\rho_{B,j}\mapsto\sum_{ij}p_{ij}\ \rho_{A,i}\otimes\rho_{B,j}^T\ , \tag{9}$$

which is a convex combination of density operators and therefore a density operator.

(b) The Bell states are a basis for two qubits, so that

$$I\otimes I=|\phi^+\rangle\langle\phi^+|+|\psi^+\rangle\langle\psi^+|+|\phi^-\rangle\langle\phi^-|+|\psi^-\rangle\langle\psi^-|\ , \tag{10}$$

and

$$\rho_F=\frac{1-F}{3}\left(I\otimes I\right)+\left(F-\frac{1-F}{3}\right)|\phi^+\rangle\langle\phi^+|\ . \tag{11}$$

Thus $1-F=\frac{3}{4}(1-\lambda)$, or $\lambda=\frac{4}{3}F-\frac{1}{3}$.

(c) We know that

$$I\otimes T:|\phi^+\rangle\langle\phi^+|\mapsto\frac{1}{2}\mathrm{SWAP}\ . \tag{12}$$

Also the Bell states $|\phi^+\rangle$, $|\phi^-\rangle$, $|\psi^+\rangle$ are eigenstates of SWAP with eigenvalue 1 (they are invariant under interchange of the two qubits), and the Bell state $|\psi^-\rangle$ is an eigenstate with eigenvalue -1 (it is antisymmetric under interchange); therefore

$$\text{SWAP} = I \otimes I - 2|\psi^-\rangle\langle\psi^-| \ . \tag{13}$$

Thus we see that

$$I \otimes T : \rho_F \mapsto \left(\frac{1+\lambda}{4}\right) I \otimes I - \lambda|\psi^-\rangle\langle\psi^-| \ . \tag{14}$$

This operator has a negative eigenvalue for $\lambda > (1+\lambda)/4$, or $\lambda > 1/3$. Therefore ρ_F is inseparable for $\lambda > 1/3$ (that is, $F > 1/2$).

(d) In the state $I \otimes I$, $\langle \sigma \cdot a \otimes \sigma \cdot b\rangle = 0$, while in the state $|\phi^+\rangle$, $\langle \sigma \cdot a \otimes \sigma \cdot b\rangle = a \cdot b$. Therefore, in the state ρ_F, the expectation value is rescaled by the factor λ compared to the expectation value in a maximally entangled state. The correlator has expectation value $2\sqrt{2}$ in the maximally entangled state, so a violation of the CHSH inequality occurs for $\lambda > 1/\sqrt{2}$. Therefore

$$F_{\text{CHSH}} = \frac{1}{4}\left(1 + 3\ \lambda_{\text{CHSH}}\right) = \frac{1}{4}\left(1 + \frac{3}{\sqrt{2}}\right) \approx .7803 \ . \tag{15}$$

Problem 3.4

Substituting the expression for $\sqrt{p_a}|\psi_a\rangle$ into our expression for $\sqrt{r_j}|e_j\rangle$, we obtain

$$\sqrt{r_j}|e_j\rangle_{AB} = \sum_{a,\mu} V_{ja}U_{a\mu}\sqrt{s_\mu}|f_\mu\rangle_A \otimes |\varphi_a\rangle_B \ . \tag{16}$$

Therefore,

$$r_j = r_j\langle e_j|e_j\rangle = \sum_{a,b,\mu}\left[V^*_{jb}U^*_{b\mu}V_{ja}U_{a\mu}\langle\varphi_b|\varphi_a\rangle\right] s_\mu = \sum_\mu D_{j\mu}s_\mu \ . \tag{17}$$

We need to check that $D_{j\mu}$ is doubly stochastic.

First note that $D_{j\mu} = \langle v_{j\mu}|v_{j\mu}\rangle$, where $|v_{j\mu}\rangle = \sum_a U_{a\mu}V_{ja}|\varphi_a\rangle$; thus, $D_{j\mu}$ is real and nonnegative. Furthermore,

$$\sum_\mu D_{j\mu} = \sum_{ab}\left(\sum_\mu U_{au}U^\dagger_{\mu b}\right) V_{ja}V^\dagger_{bj}\langle\varphi_a|\varphi_b\rangle = \sum_a V_{ja}V^\dagger_{aj} = 1 \ , \tag{18}$$

and

$$\sum_j D_{j\mu} = \sum_{a.b} U^\dagger_{\mu b}U_{a\mu}\left(\sum_j V^\dagger_{bj}V_{ja}\right)\langle\varphi_b|\varphi_a\rangle = \sum_a U^\dagger_{\mu a}U_{a\mu} = 1 \ . \tag{19}$$

Therefore, $D_{j\mu}$ is doubly stochastic.