Ph219/CS219 Problem Set 6

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6.1 Fault-tolerant error correction by teleportation

(a) We can write $\bar{X}_a$ as a product of X_α's and Z_β's:

$$\bar{X}_a = (\mathrm{sgn}) \left(\bigotimes_{\alpha=1}^{n} X_\alpha^{u_\alpha} \right) \left(\bigotimes_{\beta=1}^{n} Z_\beta^{v_\beta} \right), \quad u_\alpha, v_\beta \in \{0,1\}.$$

Then, $\bar{X}_{A,a} \otimes \bar{X}_{B,a}$ can be written as

$$\bar{X}_{A,a} \otimes \bar{X}_{B,a} = \left(\bigotimes_{\alpha=1}^{n} (X_{A,\alpha} \otimes X_{B,\alpha})^{u_\alpha} \right) \left(\bigotimes_{\beta=1}^{n} (Z_{A,\beta} \otimes Z_{B,\beta})^{v_\beta} \right). \quad \text{(S1)}$$

Therefore, if we can do bitwise Bell measurements, $\bar{X}_{A,a} \otimes \bar{X}_{B,a}$ is given by the product (parity) of all the $X_{A,\alpha} \otimes X_{B,\alpha}$ and $Z_{A,\beta} \otimes Z_{B,\beta}$ outcomes for which $u_\alpha, v_\beta = 1$ in (S1). In a similar way, we can get the values of $\bar{Z}_{A,a} \otimes \bar{Z}_{B,a}$.

(b) Consider the stabilizer generators for the two code blocks of the form $M_i \otimes M_i$. These generators are symmetric between the two blocks and hence can be written as a product of $X_{A,\alpha} \otimes X_{B,\alpha}$ and $Z_{A,\alpha} \otimes Z_{B,\alpha}$. By taking products of the appropriate bitwise Bell measurement outcomes (similar to part (a)), we can figure out the outcome of measuring $M_i \otimes M_i$.

When an error that anticommutes with M_i occurs, we get -1 on measuring M_i. For $M_i \otimes M_i$, we get -1 when that error occurs in either block A or block B, but not when it occurs in both simultaneously (i.e. same error acting on the same qubits in each block). Therefore, the set of stabilizer generators $\{M_i \otimes M_i\}$ allows us to correct all the errors (up to t of them) that the original single-block code can correct, except that the error can now occur in either block, but not both simultaneously[1]. Note that you cannot identify which block the error occurred in.

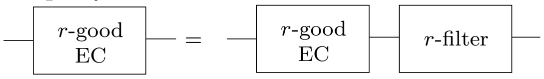
The simultaneous error does not matter, because it does not change the value of the bitwise Bell measurements. Being unable to identify which

[1] For example, for a CSS code with $t = 1$, the symmetrized generator set can correct $X_{A,\alpha} \otimes \mathbb{1}_{B,\alpha}$ and $X_{A,\alpha} \otimes Z_{B,\alpha}$, but not $X_{A,\alpha} \otimes X_{B,\alpha}$.

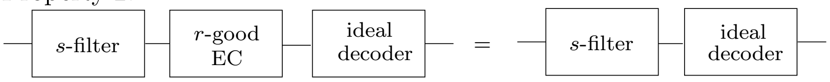
block the error occurred in is also of no consequence because, whether it was in block A or block B, the error flips the same bitwise Bell measurement outcomes. To correct for it, we use the symmetrized generators to identify which qubits had errors, and then flip the bitwise Bell measurement results affected by those errors.

(c) We want to show that the given circuit satisfies the following two properties:

Property 1:



Property 2:



We are given that the state preparation circuit satisfies the property shown in Figure 1. Furthermore, from class, we know that we can perform Pauli gates fault-tolerantly using any stabilizer code, i.e. the encoded Pauli operation circuit satisfies the property shown in Figure 2.

Figure 1: Encoded Bell state preparation circuit - Property A

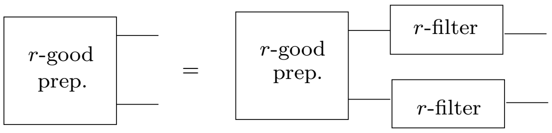
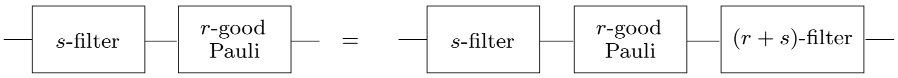


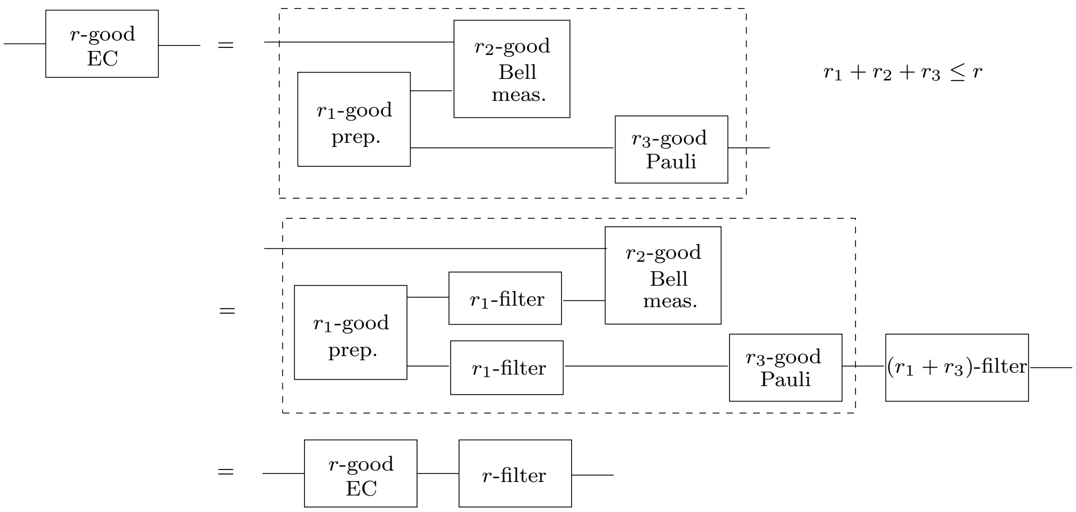
Figure 2: Encoded Pauli gate - Property B



Given properties A and B, and assuming that given any set of bitwise Bell measurement results, the classical processing always tells us a particular Pauli gate to apply (can be chosen by convention for uncorrectable outcomes), we can easily show Property 1 (see Figure 3).

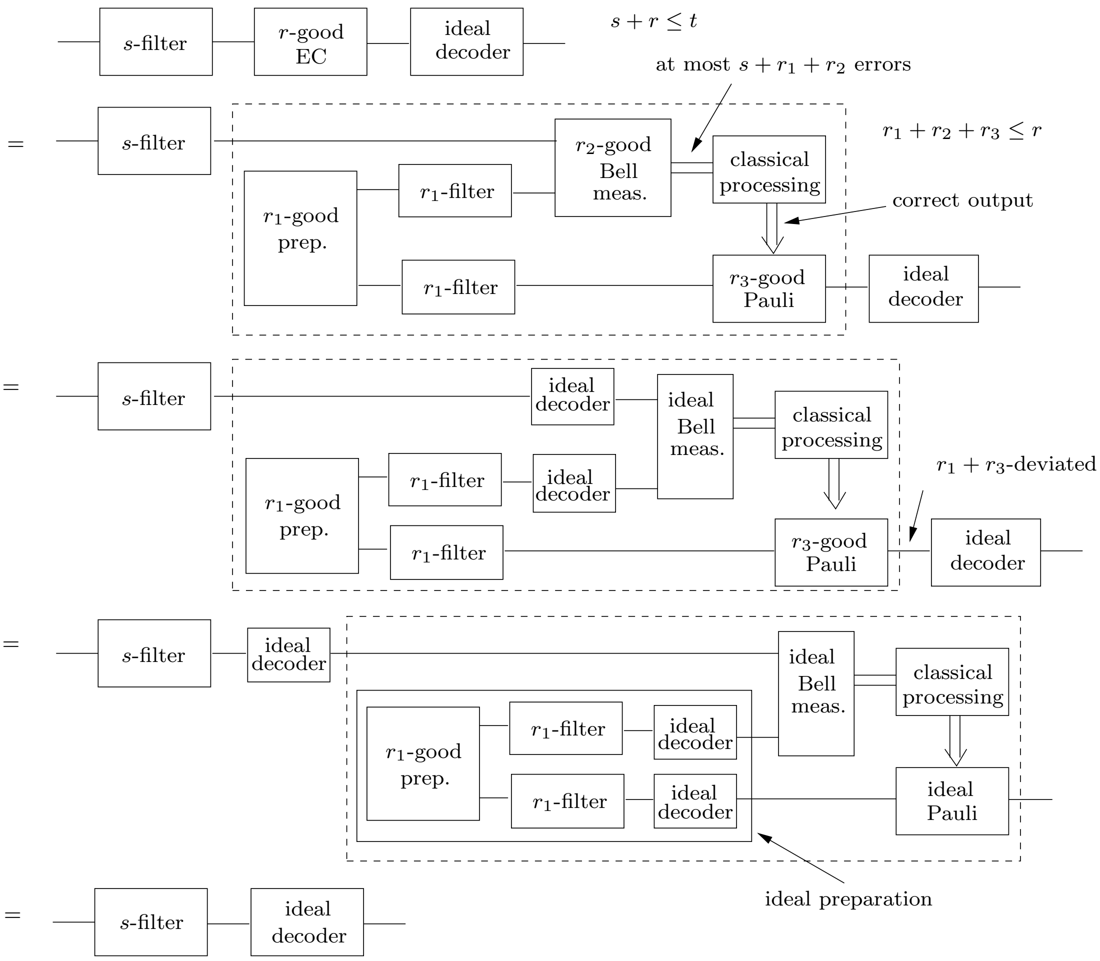
Property 2 is shown in Figure 4. Because the bitwise Bell measurements are done transversally between the two input blocks, the output from the Bell measurement circuit has at most $s+r_1+r_2$ errors. We have explicitly included the classical processing to be done after the Bell measurement.

Figure 3: Property 1



As usual, this classical processing is assumed to be ideal. The critical step in the argument is this: with at most $s + r_1 + r_2 \leq t$ errors in the bitwise Bell measurement outcomes, the classical processing can correct these errors, and give the correct output. This is then use to determine the correct Pauli gate to apply to complete the teleportation.

Figure 4: Property 2



6.2 Fault-tolerant error correction without ancilla verification

If we have a state $|\phi\rangle$ stabilized by generators $\{M_i\}$, then $C|\phi\rangle = CM_i|\phi\rangle = (CM_iC^\dagger)C|\phi\rangle$, i.e. we get a state stabilized by $\{CM_iC^\dagger\}$, where C denotes a CNOT gate. Therefore, we can understand the action of a CNOT gate on a given state by its action by conjugation on the stabilizer generators.

Let us work through the entire circuit, assuming no faults, beginning with the initial state and propagating it through the circuit. This will answer both parts (a) and (b). Notation: C_{ij} represents a CNOT gate with ith qubit as control, jth qubit as target.

1. Initial state: The initial state of the four ancilla qubits is $|0+00\rangle$, which can equivalently be represented by the stabilizer $\{ZIII, IXII, IIZI, IIIZ\}$.

2. Encoding circuit: The encoding circuit consists of three CNOT gates, C_{23} followed by C_{21} and C_{34}:

$$\begin{array}{lclclcl} ZIII & \xrightarrow{C_{23}} & ZIII & \xrightarrow{C_{21}} & ZZII & \xrightarrow{C_{34}} & ZZII \\ IXII & \longrightarrow & IXXI & \longrightarrow & XXXI & \longrightarrow & XXXX \\ IIZI & \longrightarrow & IZZI & \longrightarrow & IZZI & \longrightarrow & IZZI \\ IIIZ & \longrightarrow & IIIZ & \longrightarrow & IIIZ & \longrightarrow & IIZZ. \end{array}$$

   We end up with the state stabilized by $\{ZZII, XXXX, IZZI, IIZZ\}$ which is easily recognized to be the cat state $\frac{1}{\sqrt{2}}(|0000\rangle+|1111\rangle)$, by noting its invariance under simultanous 4-qubit bit-flips, and under simultanous pairwise phase-flips.

3. Interaction circuit: The data is stabilized by either $\{IIII, XXXX\}$ or $\{IIII, -XXXX\}$[2]. The data-ancilla (product) state before entering the interaction circuit is given by $\{IIII, \pm XXXX\}\otimes\{ZZII, XXXX, IZZI, IIZZ\}$. The interaction circuit acts non-trivially only on $\pm XXXX \otimes XXXX$ and $IIII \otimes XXXX$:

$$\begin{aligned} \pm XXXX \otimes XXXX &\longrightarrow IIII \otimes \pm XXXX, \\ IIII \otimes XXXX &\longrightarrow XXXX \otimes XXXX \end{aligned}$$

   Observe that the post-interaction data-ancilla state remains a product state (as must be the case for the interaction to not affect the data qubits): $\{IIII, \pm XXXX\} \otimes \{ZZII, \pm XXXX, IZZI, IIZZ\}$. This means that we can just deal with the ancilla state after this.

4. Decoding circuit:

$$\begin{array}{lclclcl} ZZII & \xrightarrow{C_{24}} & ZZII & \xrightarrow{C_{31}} & ZZZI & \xrightarrow{C_{23}} & ZIZI \\ \pm XXXX & \longrightarrow & \pm XXXI & \longrightarrow & \pm IXXI & \longrightarrow & \pm IXII \\ IZZI & \longrightarrow & IZZI & \longrightarrow & IZZI & \longrightarrow & IIZI \\ IIZZ & \longrightarrow & IZZZ & \longrightarrow & IZZZ & \longrightarrow & IIZZ. \end{array}$$

[2] $\pm XXXX$ is the only relevant stabilizer generator here, and its square gives the identity. We need both $\pm XXXX$ and $IIII$ for the data qubits so that, when combined with the stabilizer for the ancilla, we get a full set of generators for the combined stabilizer.

Therefore, the final ancilla state is stabilized by $\{ZIZI, \pm IXII, IIZI, IIZZ\} \equiv \{ZIII, \pm IXII, IIZI, IIIZ\}$, where in the last equality, we have taken products of the generators to get simpler ones.

5. Measurements: Suppose we want to measure an operator O that either commutes with all the stabilizer generators of the state or anticommutes with all of them. Such a measurement will give a definite result (i.e. ± 1 with probability 1). If $\pm O$ is in the stabilizer, the measurement result will be ± 1. Therefore, if we measure Z_1, X_2, Z_3 and Z_4 on the ancilla state, we will get:
$$Z_1 = 1, \quad X_2 = \pm 1, \quad Z_3 = 1, \quad Z_4 = 1$$

(a) From the post-interaction state above, we see that, indeed, the eigenvalue of the $XXXX$ stabilizer on the data qubits is equal to the eigenvalue of $XXXX$ on the post-interaction ancilla state. If there are no faults, the sign of the eigenvalue is carried through the decoding circuit to the stabilizer operator $\pm IXII$ of the final state. An X measurement on qubit 2 of the ancilla will thus yield the desired result.

(b) As shown above, in the absence of faults, the measurements will yield $Z_1 = 1, Z_3 = 1$ and $Z_4 = 1$.

(c) First, consider faults in the encoding circuit that cause an error on a single qubit in the ancilla. These can arise due to faulty preparation of a qubit or from a faulty CNOT causing only a single qubit error. Observe that qubits 1 and 4 of the ancilla occur only as targets of CNOT gates, and hence, a single error in either qubit will not propagate into a weight-two error. A single fault in qubit 2 *before* the first CNOT gate (C_{23}) will propagate into a weight-four error $XXXX$ on the output of the encoding circuit, which is equivalent to no error[3]. On the other hand, a single fault in qubit 2 *after* C_{23} (due to, for example, a faulty C_{23}) will propagate only into the first qubit, hence giving the weight-two error $XXII$. A single fault in qubit 3, either due to preparation or because of a faulty C_{23} will propagate only into the fourth qubit, hence giving the weight-two error $IIXX$. Other single qubit faults do not give rise to a weight-two X error.

A faulty CNOT gate can also cause both its target and control qubits to have errors simultaneously. A faulty C_{23} will in this case cause a weight-four error on the output, which as before is no error. A faulty C_{21} causes the weight-two error $XXII$, and a faulty C_{34} causes the error $IIXX$.

Therefore, the only possible weight-two errors are $XXII$ and $IIXX$.

(d) Suppose the ancilla has a weight-one X error, which can occur in any one of the four qubits, after the interaction circuit. The simplest way to figure out the effect of these errors on the output ancilla state is to propagate the

[3] Note that the cat state has $XXXX$ as one of its stabilizer operators and therefore, it cannot have more than weight-two X errors.

errors through the decoding circuit. The propagated errors then act by conjugation on the stabilizer operators of the original (ideal) final ancilla state to give the stabilizer of the new state. We will do this for each possible error, and the no-fault final state has stabilizer, as found before, $\{ZIII, \pm IXII, IIZI, IIIZ\}$.

Error			Meas. results		
Before decoder	After decoder	New stabilizer	Z_1	Z_3	Z_4
$XIII$	$XIII$	$\{(-Z)III, \pm IXII, IIZI, IIIZ\}$	-1	1	1
$IXII$	$IXXX$	$\{ZIII, \pm IXII, II(-Z)I, III(-Z)\}$	1	-1	-1
IIX	$XIXI$	$\{(-Z)III, \pm IXII, II(-Z)I, IIIZ\}$	-1	-1	1
$IIIX$	$IIIX$	$\{ZIII, \pm IXII, IIZI, III(-Z)\}$	1	1	-1

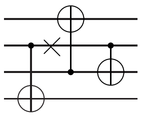
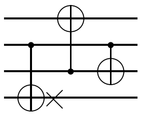
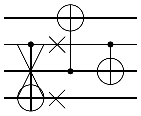
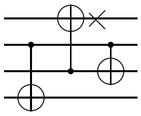
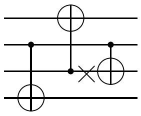
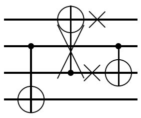
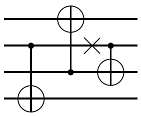
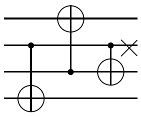
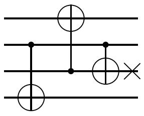
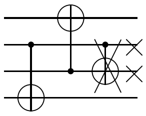
From this, we see that the only possible syndromes (i.e. Z measurement outcomes) for a weight-one error occuring just before the decoding circuit are $(Z_1, Z_3, Z_4) = (-1, 1, 1), (1, -1, -1), (-1, -1, 1)$ and $(1, 1, -1)$.

Suppose the cat state, on exiting the encoding circuit, has a weight-two X error. From part (c), we know that the only weight-two errors that can arise are $XXII$ and $IIXX$. We repeat the above analysis for these two errors, propagating them through the interaction and the decoding circuits.

Error			Meas. results		
After encoder	After decoder	New stabilizer	Z_1	Z_3	Z_4
$XXII$	$XXXX$	$\{(-Z)III, \pm IXII, II(-Z)I, III(-Z)\}$	-1	-1	-1
$IIXX$	$XIXX$	$\{(-Z)III, \pm IXII, II(-Z)I, III(-Z)\}$	-1	-1	-1

We see that both weight-two errors give rise to the same syndrome: $(Z_1, Z_3, Z_4) = (-1, -1, -1)$. This is different from any of the weight-one error syndromes found before, and also different from the weight-zero syndrome $(1, 1, 1)$.

(e) Consider a single fault in the decoding circuit. Note that the case of a single fault occuring before the first CNOT gate of the decoding circuit was already handled in part (d), so we only need to consider faults occuring after the first CNOT. We will identify all possible locations of the single fault in the decoding circuit and do a similar analysis as done in part(d). (I will omit writing down the new stabilizer operators.)

Error			Meas. results		
Location	Error operator	After decoder	Z_1	Z_3	Z_4
	$IXII$	$IXXI$	1	-1	1
	$IIIX$	$IIIX$	1	1	-1
	$IXIX$	$IXXX$	1	-1	-1
	$XIII$	$XIII$	-1	1	1
	$IIXI$	$IIXI$	1	-1	1
	$XIXI$	$XIXI$	-1	-1	1
	$IXII$	$IXXI$	1	-1	1
	$IXII$	$IXII$	1	1	1
	$IIXI$	$IIXI$	1	-1	1
	$IXXI$	$IXXI$	1	-1	1

A single fault occuring in the measurement circuit (and no where else) will result in flipping one of the three ideal Z measurement results, yielding

$(Z_1, Z_3, Z_4) = (-1, 1, 1), (1, -1, 1)$ or $(1, 1, -1)$.

For fault-tolerance, we need to show that a single fault cannot cause two errors in the data. Having two errors in the data will cause the next round of (ideal) error correction to mistake it as a single error and do a recovery that may result in a logical error. Since the interaction with the data in our circuit is transversal, the only faults that can cause two errors in the data are those in the encoder that result in a weight-two X error in the ancilla, which will then be propagated to two data qubits. Therefore, our syndrome extraction circuit has to be able to correct for this in order to be fault-tolerant.

Now, notice that none of the single fault locations in the decoding circuit or the measurement circuit results in a syndrome that is identical to that of a weight-two X error (from a single fault in the encoder). Similarly for weight-one X errors in the interaction or encoding circuits, as stated in part (d). This means that we get the syndrome $(-1, -1, -1)$ if and only if a weight-two error occurred in the encoder. Getting this syndrome thus tells us that a weight-two error of the form $XXII$ or $IIXX$ has been propagated to the data, and we need to do a recovery. The recovery operation is the same for both errors, since $\pm XXXX$ is in the stabilizer of the data, i.e. we just need to apply $XXII$ (or $IIXX$) to the data qubits. Hence, our circuit is fault-tolerant.

Note that Z errors in the syndrome extraction circuit do not matter as these will not affect the syndromes (although they will affect the X measurement result, which has to be checked by repeating the measurement). Nor will they propagate through the interaction circuit to cause errors in the data.

6.3 Fault-tolerant error correction via gauge qubit measurement

(a) Consider the stabilizer generator $Z_{\text{col}-1}Z_{\text{col}-2}$. We can rewrite it as

$$Z_{\text{col}-1}Z_{\text{col}-2} = (Z_1Z_4Z_7)(Z_2Z_5Z_8) = (Z_1Z_2)(Z_4Z_5)(Z_7Z_8),$$

where the expression on the right is a product of three gauge operators (pairs of Z's on neighboring qubits in the same row). In the same way, all other stabilizer generators can each be written as a product of three gauge operators. Therefore, if we can measure all the gauge operators without faults[4], the measurement outcome of each stabilizer will be given by the product of the respective outcomes of the gauge measurements (eg. $Z_{\text{col}-1}Z_{\text{col}-2}$ = product of Z_1Z_2, Z_4Z_5 and Z_7Z_8 measurement outcomes).

[4] Note that the gauge operators commute with the stabilizer and logical operators. Therefore, measuring them will not affect the encoded state.

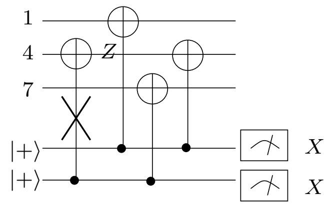
The recovery procedure for X errors is given in the following table. The recovery for Z errors is carried out in the same way, with $X \leftrightarrow Z$ and row $\leftrightarrow$ column.

Syndrome		X error	Recovery
$Z_{\mathrm{col}-1}Z_{\mathrm{col}-2}$	$Z_{\mathrm{col}-2}Z_{\mathrm{col}-3}$		
+1	+1	no error	no recovery needed
+1	-1	error in col 3	apply X to any qubit in col 3
-1	+1	error in col 1	apply X to any qubit in col 1
-1	-1	error in col 2	apply X to any qubit in col 2

Notice that the recovery operation can be applied to any one of the three qubits in the column with the X error. This is because, even if we apply X to a qubit different from the one that actually suffered an X error, the overall action is a gauge operator, which has no effect on the encoded space.[5]

(b) Consider the circuit for $j = 1$. Suppose we omit the third measurement and measure only X_1X_4 and X_4X_7 (see Figure 5). Furthermore, suppose that the first CNOT gate is faulty and causes a Z error on the target qubit (qubit 4).

Figure 5: Omit redundant measurement.



The error propagates through the circuit and causes the X_1X_4 measurement result to be flipped, and hence flipping the value of $X_{\mathrm{row}-1}X_{\mathrm{row}-2}$. Assuming that the input has no error, we will then conclude (wrongly) that there was an error in row 1, and perform a recovery Z on say qubit 1. The overall effect is the operator Z_1Z_4, which is *not* a gauge operator. In the next cycle of (ideal) error correction, this weight-two error will flip the measurement outcome of $X_{\mathrm{row}-2}X_{\mathrm{row}-3}$, and lead us to misinterpret a Z error in row 3. If we then try to recover by applying a Z on qubit 7, we would have committed a logical error $\bar{Z} = Z_1Z_4Z_7$.

With redundant measurement, the situation described above will not happen. Consider the same fault and error. With an additional X_1X_7 mea-

[5] You may be curious why this code has $n = 9$ and $k = 1$, but only has 4 stabilizer generators instead of $n - k = 9 - 1 = 8$ of them. This is precisely because of the gauge freedom. If instead we specify a particular gauge by choosing, say, all the Z-Z gauge operators to be +1, we will get the 8 generators given in class.

surement, we can check for consistency of the syndrome measurement results, by taking the product of all three results. If we end up with $+1$, we know the syndromes are correct because $X_1X_7 = (X_1X_4)(X_4X_7)$ (note that, because of the structure of the circuit, a single fault cannot result in two wrong measurement outcomes). If we end up with -1, we know that one of the measurement results is wrong. In this case, we do not do anything. The overall effect of the faulty CNOT is then just a single Z on qubit 4. This will be corrected in the next (ideal) error correction cycle.

Is this procedure fault-tolerant? For property 1 (see Problem 1), since $t = 1$ here, we only have two cases: $r = 0$ and $r = 1$. Observe that every state in the Hilbert space passes (unchanged) through a 1-filter, where the 1-filter in this case is defined so that any state equal to a state in the codespace with a single non-gauge Pauli operator applied, passes through it[6]. Therefore, the $r = 1$ case is trivially true. For $r = 0$, i.e. ideal EC, every input state will be interpreted to have at most a single error and returned to the codespace (although it may end up in the wrong encoded state), and hence passes through the 0-filter. Hence, the syndrome measurement satisfies property 1.

For property 2, we again only have two cases: $s = 1, r = 0$ and $s = 0, r = 1$. The $s = 1, r = 0$ case is trivially true - it is just a statement that the ideal EC corrects 1 error without fault. For $s = 0, r = 1$, we need to check that each possible single fault in the syndrome measurement causes at most one error in the data. All the syndrome measurement circuits have the CNOT gates arranged such that, if a fault occurs before the measurements, either only one data qubit suffers an error, or two data qubits simultaneously suffer a weight-two error that is one of the gauge operators, which is actually not an error. The redundancy in the final measurements then ensure that we do not cause more errors by doing a wrong recovery. Therefore, property 2 holds.

(c) Each of the circuits measuring the gauge operators in the same row or column has 3 qubit preparations, 6 CNOT gates and 3 measurements, giving a total of 12 locations. There are six such circuits (three rows, three columns), so there are altogether $6 \times 12 = 72$ locations.

[6] Gauge operators can be applied any number of times, since they do not affect the syndrome and thus does not prevent the (ideal) EC from correcting any non-gauge Pauli error.