Ph219/CS219 Problem Set 7

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Problem 1: Fusion of S_3 charges.

Fusing the first pair of charges gives $[2] \times [2] = [+] + [-] + [2]$. Similarly for fusing the other pair. Fusing the output particles of the two fusions can give a singlet $[+]$ in 3 different ways:

$$[+] \times [+] = [+], \quad [-] \times [-] = [+], \quad [2] \times [2] = [+] + \ldots$$

Using eq. (13), we can write the singlet state $|+\rangle$ first in the terms of its two constituent particles. Then, each of these two particles can be written in terms of the pair of $[2]$ charges fused together to form it. We get:

$$\begin{aligned}
[+] \times [+] : \quad & |+\rangle \mapsto |+\rangle \otimes |+\rangle \\
& \mapsto |\xi_{++}\rangle = \frac{1}{2}\left(|2_+, 2_-\rangle + |2_-, 2_+\rangle\right) \otimes \left(|2_+, 2_-\rangle + |2_-, 2_+\rangle\right), \\
[-] \times [-] : \quad & |+\rangle \mapsto |-\rangle \otimes |-\rangle \\
& \mapsto |\xi_{--}\rangle = \frac{1}{2}\left(|2_+, 2_-\rangle - |2_-, 2_+\rangle\right) \otimes \left(|2_+, 2_-\rangle - |2_-, 2_+\rangle\right), \\
[2] \times [2] : \quad & |+\rangle \mapsto \frac{1}{\sqrt{2}}\left(|2_+, 2_-\rangle + |2_-, 2_+\rangle\right) \\
& \mapsto |\xi_{22}\rangle = \frac{1}{\sqrt{2}}\left(|2_-, 2_-, 2_+, 2_+\rangle + |2_+, 2_+, 2_-, 2_-\rangle\right).
\end{aligned}$$

Problem 2: Flavor and color.

(a) Let us start with working out the states in the multiplet $\mathcal{L}_{[C_3]} \otimes \mathcal{L}_{[C_3]}$. The irrep $[C_3]$ (i.e. flux in C_3 and trivial charge) has two basis vectors $|(123)\rangle$ and $|(132)\rangle$. Hence there are four flavor-color states in $\mathcal{L}_{[C_3]} \otimes \mathcal{L}_{[C_3]}$: $|(123)\rangle \otimes |(123)\rangle$, $|(123)\rangle \otimes |(132)\rangle$, $|(132)\rangle \otimes |(123)\rangle$ and $|(132)\rangle \otimes |(132)\rangle$. Let us figure out how to write the first state $|(123)\rangle \otimes |(123)\rangle$.

The basis vector $|(123)\rangle$ transforms according to eq. (7): $A^\mu|(123)\rangle = |(123)\rangle$, $\quad A^\sigma|(123)\rangle = |(132)\rangle$, where $\mu = (123)$ and $\sigma = (23)$ as defined in the question[1]. B^h acts trivially: $B^h|(123)\rangle = \delta_{h,(123)}|(123)\rangle$. Therefore, the state $|(123)\rangle\otimes|(123)\rangle$ transforms under A^μ_{fl} and A^μ_{col} as (we will disscuss A^σ transformations later):

$$A^\mu_{\text{fl}}(|(123)\rangle \otimes |(123)\rangle) = (A^\mu|(123)\rangle) \otimes |(123)\rangle = |(123)\rangle) \otimes |(123)\rangle$$
$$A^\mu_{\text{col}}(|(123)\rangle \otimes |(123)\rangle) = |(123)\rangle \otimes (A^\mu|(123)\rangle) = |(123)\rangle) \otimes |(123)\rangle$$

For B^h, the state is invariant under $B^{(123)}_{\text{fl,col}}$ and 0 otherwise.

We need to construct a superposition of $|z, w\rangle$ that transforms in the same way. Recall that the states $|z, w\rangle$ transform under the flavor and color operators according to equations (3) and (4). The transformation of $|(123)\rangle \otimes |(123)\rangle$ under B^h_{col} restricts the value of w to $w = (123)$. Transformation under B^h_{fl} further requires $zwz^{-1} = (123)^{-1} = (132)$ (according to eq. (3)). The only possible values of z are hence the two-cycles (12), (23) and (31). Since the two-cycles transform among themselves under μ acting on the left (for flavor) and μ^{-1} acting on the right (for color), we see that we should take their symmetric superposition:

$$|(123)\rangle \otimes |(123)\rangle \mapsto \frac{1}{\sqrt{3}} (|(12), (123)\rangle + |(23), (123)\rangle + |(31), (123)\rangle) .$$

The transformation properties under A^σ then serve to define the appropriate superpositions for the other three flavor-color states. Applying A^σ_{col} on both sides gives

$$|(123)\rangle) \otimes |(132)\rangle \mapsto \frac{1}{\sqrt{3}} (|(123), (132)\rangle + |e, (132)\rangle + |(132), (132)\rangle) .$$

Let us check that this transforms correctly. The left-hand side is invariant under A^μ_{fl} and A^μ_{col}. Applying A^μ_{fl} or A^μ_{col} on the right hand side just permutes the three $|z, w\rangle$ terms, which also leaves the state invariant. Thus we have the correct superposition.

Applying A^σ_{fl} to $|(123)\rangle \otimes |(123)\rangle$ gives

$$|(132)\rangle \otimes |(123)\rangle \mapsto \frac{1}{\sqrt{3}} (|(132), (123)\rangle + |e, (123)\rangle + |(123), (123)\rangle) .$$

Finally, applying $A^\sigma_{\text{fl}} \otimes A^\sigma_{\text{col}}$ to $|(123)\rangle \otimes |(123)\rangle$ gives (note that the flavor and color operators on $|z, w\rangle$ commute)

$$|(132)\rangle) \otimes |(132)\rangle \mapsto \frac{1}{\sqrt{3}} (|(31), (132)\rangle + |(23), (132)\rangle + |(12), (132)\rangle) .$$

You can check that they transform correctly.

[1] Note that μ and σ generate the whole group S_3. Furthermore, since $A^{g_1 g_2} = A^{g_1} A^{g_2}$, it is sufficient to consider only A^μ and A^σ.

Next, let us consider the multiplet $\mathcal{L}_{[C_3,\bar{\omega}]} \otimes \mathcal{L}_{[C_3,\omega]}$. We again have four flavor-color states: $|(123),\bar{\omega}\rangle \otimes |(123),\omega\rangle$, etc. The basis vectors for $[C_3,\omega]$ transform according to equations (14) and (15). For the conjugate representation $[C_3,\bar{\omega}]$, $|(123),\bar{\omega}\rangle$ transforms under A^g and B^h as $|(123),\omega\rangle$, and $|(132),\bar{\omega}\rangle$ as $|(132),\omega\rangle$. Notice that the basis states transform under B^h in the same way as for the states in $[C_3]$. Therefore, we get the same conclusion as before that the state $|(123),\bar{\omega}\rangle \otimes |(123),\omega\rangle$ must be built from superposition of the $|z,w\rangle$ states $|(12),(123)\rangle$, $|(23),(123)\rangle$ and $|(31),(123)\rangle$. Transformations under A^μ according to eq. (14) then tell us that the appropriate superposition is

$$|(123),\bar{\omega}\rangle \otimes |(123),\omega\rangle \mapsto \frac{1}{\sqrt{3}}(|(12),(123)\rangle + \omega|(23),(123)\rangle + \bar{\omega}|(31),(123)\rangle).$$

Applying $A^\sigma_{\text{fl,col}}$ gives the other three states:

$$\begin{aligned}
|(123),\bar{\omega}\rangle \otimes |(132),\omega\rangle &\mapsto \frac{1}{\sqrt{3}}(|(123),(132)\rangle + \omega|e,(132)\rangle + \bar{\omega}|(132),(132)\rangle),\\
|(132),\bar{\omega}\rangle \otimes |(123),\omega\rangle &\mapsto \frac{1}{\sqrt{3}}(|(132),(123)\rangle + \omega|e,(123)\rangle + \bar{\omega}|(123),(123)\rangle),\\
|(132),\bar{\omega}\rangle \otimes |(132),\omega\rangle &\mapsto \frac{1}{\sqrt{3}}(|(31),(132)\rangle + \omega|(23),(132)\rangle + \bar{\omega}|(12),(132)\rangle).
\end{aligned}$$

We also want the pure vortices $|w\rangle = \frac{1}{\sqrt{|S_3|}}\sum_{z\in S_3}|z,w\rangle$, $w \in C_3$ in the $\mathcal{L}_{[C_3]} \otimes \mathcal{L}_{[C_3]}$ multiplet. The pure vortex $|(123)\rangle$ must be given by a superposition of the flavor-color states $|(123)\rangle \otimes |(123)\rangle$ and $|(132)\rangle \otimes |(123)\rangle$ only, since the other two states have $w = (132)$ on the right-hand side. The appropriate superposition is

$$|(123)\rangle = \frac{1}{\sqrt{6}}\sum_{z\in S_3}|z,(123)\rangle \mapsto \frac{1}{\sqrt{2}}\left(|(123)\rangle \otimes |(123)\rangle + |(132)\rangle \otimes |(123)\rangle\right).$$

The pure vortex $|(132)\rangle$ must be built out of the other two states:

$$|(132)\rangle = \frac{1}{\sqrt{6}}\sum_{z\in S_3}|z,(132)\rangle \mapsto \frac{1}{\sqrt{2}}\left(|(123)\rangle \otimes |(132)\rangle + |(132)\rangle \otimes |(132)\rangle\right).$$

(b) We want to build a color singlet from a pair of charges with flavor $|2_+\rangle$. Using equation (13), we know that we can build a color singlet by taking the superposition $\frac{1}{\sqrt{2}}(|2_+,2_-\rangle + |2_-,2_+\rangle)$ for the color states of the two particles. Therefore, the overall singlet state in the flavor-color basis $|\text{flavor}_1,\text{color}_1;\text{flavor}_2,\text{color}_2\rangle$ is simply

$$|\psi\rangle := \frac{1}{\sqrt{2}}\left(|2_+,2_+;2_+,2_-\rangle + |2_+,2_-;2_+,2_+\rangle\right), \tag{S1}$$

so that the color states combine correctly to form $|+\rangle$.

To get the color singlet $|\psi\rangle$ in the $|z_1, w_1; z_2, w_2\rangle$ basis, we use the following two equations, given in the question (right side is written in terms of $|z\rangle$):

$$\underbrace{|2_+\rangle}_{\text{flavor}_i} \otimes \underbrace{|2_+\rangle}_{\text{color}_i} \mapsto \frac{1}{\sqrt{3}}(|\sigma^{-1}\rangle + \bar{\omega}|\mu\sigma^{-1}\rangle + \omega|\mu^{-1}\sigma^{-1}\rangle)$$

$$= \frac{1}{\sqrt{3}}(|(23)\rangle + \bar{\omega}|(12)\rangle + \omega|(31)\rangle)$$

$$|2_+\rangle \otimes |2_-\rangle \mapsto \frac{1}{\sqrt{3}}(|e\rangle + \bar{\omega}|\mu\rangle + \omega|\mu^{-1}\rangle)$$

$$= \frac{1}{\sqrt{3}}(|e\rangle + \bar{\omega}|(123)\rangle + \omega|(132)\rangle).$$

Inserting these into eq. (S1) immediately gives $|\psi\rangle$ in the $|z_1, w_1; z_2, w_2\rangle$ basis with $w_1 = w_2 = e$ (charges).

The neutrality conditions are easy to check: $w_1 w_2 = e$, so we always have $B^h_{\text{global}}|\psi\rangle = \delta_{h,1}|\psi\rangle$. Furthermore, A^g_{global} acts like the color operators A^g_{col} on the individual $|z, w\rangle$ (see eq. (9)), and since $|\psi\rangle$ is a color singlet, it is hence also invariant under A^g_{global}. Therefore, $|\psi\rangle$ satisfies the neutrality conditions.

Problem 3: Measurement by braiding and fusion.

(a) We have vortices in the class C_3, which implies that the total flux $w := w_1 w_2 \dots w_n$ is also in C_3. We want to braid a charge around all the vortices. Recall that this braiding changes z to zw^{-1} (for the charge), which is the same action as applying A^w_{col}. Since we are dealing with color states here, the basis vectors transform according to A^g_{col} when written in the $|z\rangle$ basis, and hence, the braiding operation is exactly the group action of w on the basis vectors, as suggested in the hint.

The charges (in irrep [2]) start in the (color) singlet state $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|2_+, 2_-\rangle + |2_-, 2_+\rangle\big)$. Braiding the first charge around the vortices gives a new state

$$|\psi_w\rangle := (w \otimes e)|\psi\rangle,$$

and we want to figure out how this new state transforms.

We have three possible values of w. For $w = e$, nothing happens to the charge, so $|\psi_e\rangle = |\psi\rangle$ and the charges fuse back into [+] with probability 1. For $w = (123)$, $|\psi_{(123)}\rangle = \frac{1}{\sqrt{2}}(\omega|2_+, 2_-\rangle + \bar{\omega}|2_-, 2_+\rangle)$. The probability of the fusion outcome [+] is $|\langle +|\psi_{(123)}\rangle|^2 = \frac{1}{4}$, where we have used the expression for $|+\rangle$ within $[2] \times [2]$ (eq. (13)). The probability for [−] is $|\langle -|\psi_{(123)}\rangle|^2 = \frac{3}{4}$, and hence the probability for [2] is 0. Similarly, for $w = (132)$, $|\psi_{(132)}\rangle = \frac{1}{\sqrt{2}}\big(\bar{\omega}|2_+, 2_-\rangle + \omega|2_-, 2_+\rangle\big)$ and the outcome probabilities are exactly the same as those for $w = (123)$. Notice that we never get the outcome [2].

(b) Bringing the vortex around the charge is equivalent to bringing the charge around the vortex (in the same direction). As explained in part (a), this braiding operation is just the group operation on the basis vector for the charge.

Suppose we have a trivial charge, i.e. in state $|+\rangle$. Then, since $|+\rangle$ is invariant under all group elements, it is invariant under the braiding. We hence end up with the vortices being in the original $|\eta\rangle$ state unchanged, and they will always annihilate.

Suppose the charge is in the irrep [2], and furthermore, suppose it is in the state $|2_+\rangle$. Then, given the vortices in state $|\eta\rangle$,

$$\begin{aligned}
&|2_+\rangle \otimes |\eta\rangle \\
\xrightarrow{\text{braid}}\ &\frac{1}{\sqrt{3}}\left((12)|2_+\rangle \otimes |(12),(12)\rangle + (23)|2_+\rangle \otimes |(23),(23)\rangle + (31)|2_+\rangle \otimes |(31),(31)\rangle\right) \\
&= |2_-\rangle \otimes \underbrace{\frac{1}{\sqrt{3}}\left(\bar{\omega}|(12),(12)\rangle + |(23),(23)\rangle + \omega|(31),(31)\rangle\right)}_{=:|\eta'_+\rangle}.
\end{aligned}$$

We can do a similar analysis for a charged particle in state $|2-\rangle$:

$$|2_-\rangle \otimes |\eta\rangle \xrightarrow{\text{braid}} |2_+\rangle \otimes \underbrace{\frac{1}{\sqrt{3}}\left(\omega|(12),(12)\rangle + |(23),(23)\rangle + \bar{\omega}|(31),(31)\rangle\right)}_{=:|\eta'_-\rangle},$$

and we get the state $|\eta'_-\rangle$ that is the complex conjugate of $|\eta'_+\rangle$.

We want to know how the states $|\eta'_+\rangle$ and $|\eta'_-\rangle$ transform. Note that $|\eta'_\pm\rangle$ still describe states with trivial flux, so the relevant irreps are $[+]$, $[-]$ and $[2]$. It is easy to show that[2] $A^\mu \otimes A^\mu|\eta'_+\rangle = \omega|\eta'_+\rangle$, $A^\mu \otimes A^\mu|\eta'_-\rangle = \bar{\omega}|\eta'_-\rangle$, and B^h acts trivially. Furthermore, $A^\sigma \otimes A^\sigma|\eta_\pm\rangle = |\eta_\mp\rangle$. Therefore, $|\eta'_\pm\rangle$ transform as $|2_\pm\rangle$, i.e. they belong to the irrep [2], which corresponds to a non-trivial charge, and hence will not annihilate (to the vacuum state).[3]

(c) (*Extra credit*) Suppose the charge is originally in state $|2_+\rangle$. After braiding with the first set of vortices, the charge-vortex state is $|2_-\rangle \otimes |\eta'_+\rangle$. Braiding the charge with a vortex of a second vortex pair (making sure that the path does not loop around the first vortex pair) gives the overall state $|2_+\rangle \otimes |\eta'_+\rangle \otimes |\eta'_-\rangle$ (notation: $|\text{charge}\rangle \otimes |\text{vortex pair 1}\rangle \otimes |\text{vortex pair 2}\rangle$). From part (b), we know $|\eta'_\pm\rangle$ transform as $|2_\pm\rangle$, so the state of the remnant particles is $|2_+, 2_-\rangle$. Now, $|\langle +|2_+, 2_-\rangle|^2 = \frac{1}{2}$ (we are using a mixed

[2]The right-hand side of $|\eta'_\pm\rangle$ is written in terms of basis vectors of two vortices in the class C_2, and they have trivial charges. Hence, the operators A^g and B^h act according to eq. (7).

[3]Note that if the charge was originally in some superposition $a|2_+\rangle + b|2_-\rangle$, the [2] remnant particle will in general be entangled to the charge after the braiding: $(a|2_+\rangle + b|2_-\rangle)\otimes|\eta\rangle \xrightarrow{braid} (a|2_-\rangle \otimes |\eta'_+\rangle + b|2_+\rangle \otimes |\eta'_-\rangle)$.

notation, where the $|+\rangle$ state should really be written in the $|2_\pm\rangle$ basis according to eq. (13)), so the remnant particles fuse to a $[+]$ (annihilate, because it has trivial flux too) with probability $\frac{1}{2}$. Furthermore, $|\langle -|2_+, 2_-\rangle|^2 = \frac{1}{2}$, so we get the fusion outcome $[-]$ with probability $\frac{1}{2}$.[4]

Problem 4. Computation using S_3 anyons.

(a) Suppose we have the $|-\rangle$ state. Then, because $\langle\tilde{0}|-\rangle = 0$ and $\langle 2|-\rangle = 0$, comparing it with $|\tilde{0}\rangle$ or $|2\rangle$ always yields the answer "no". On the other hand, we can get "yes" answers if the state was $|+\rangle$, as we shall see below. Hence, if we get a "yes", we know for certain that the state is $|+\rangle$. If we always get "no" after n (large) tries, then, with high probability, the state is $|-\rangle$ (see below).

Define $Y_{\tilde{0}} := |\tilde{0}\rangle\langle\tilde{0}|$, and $N_{\tilde{0}} := \mathbb{1} - Y_{\tilde{0}} = |\tilde{1}\rangle\langle\tilde{1}| + |\tilde{2}\rangle\langle\tilde{2}|$. Similarly, define $Y_2 := |2\rangle\langle 2|$ and $N_2 := \mathbb{1} - Y_2 = |0\rangle\langle 0| + |1\rangle\langle 1|$. Observe that

$$N_{\tilde{0}}|+\rangle = \frac{1}{3\sqrt{2}}(|0\rangle + |1\rangle - 2|2\rangle), \quad N_2 N_{\tilde{0}}|+\rangle = \frac{1}{3}|+\rangle.$$

Then, $(N_2 N_{\tilde{0}})^k|+\rangle = \frac{1}{3^k}|+\rangle$, and $\|N_{\tilde{0}}|+\rangle\|^2 = \frac{1}{3}$.

Let one trial be a comparison with either $|\tilde{0}\rangle$ or $|2\rangle$ - the odd trials are comparisons with $|\tilde{0}\rangle$ and the even trials are with $|2\rangle$. Suppose we stop after n trials, all of which yielded negative answers. Then the probability that the state was originally $|+\rangle$ is given by ($k > 0$):

$$n = 2k-1 \text{ odd:} \quad \mathrm{P}(|+\rangle) = \|N_{\tilde{0}}(N_2 N_{\tilde{0}})^{k-1}|+\rangle\|^2 = \frac{\|N_{\tilde{0}}|+\rangle\|^2}{3^{2(k-1)}} = \left(\frac{1}{3}\right)^n$$

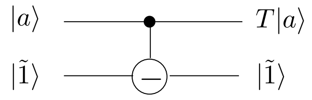
$$n = 2k \text{ even:} \quad \mathrm{P}(|+\rangle) = \|(N_2 N_{\tilde{0}})^k|+\rangle\|^2 = \frac{1}{3^{2k}} = \left(\frac{1}{3}\right)^n,$$

i.e. the probability of error (guessing $|-\rangle$ when the state was $|+\rangle$) is $(1/3)^n$ if we stop after n negative answers.

(b) Given the reference state $|\tilde{1}\rangle$, we can implement the gate $T|a\rangle = \omega^a|a\rangle$ as shown in Figure 1. Note that the $|\tilde{1}\rangle$ remains unchanged and unentangled to the input state, and can be reused.

[4] You may be worried about charge conservation. For example, in part (b), if the charged particle was in the irrep [2], you end up with a remnant particle of charge [2], but the vortices originally had with trivial charge. So where does the additional charge come from? The answer is to think of the original charged particle as half of a charge singlet (it must have been created this way). After braiding with the vortex state, the charge particle state changes as $|2_\pm\rangle \mapsto |2_\mp\rangle$, which changes the charge singlet state into a state in the irrep [2]. So there is a transfer of charge between the charges and the remnant particle. I leave you to figure out how charge conservation works out if we get $[-]$ after fusing the remnant particles in part (c).

Figure 1: T gate.



Applying $T \otimes T^2$ on two copies of $|+\rangle$ gives the state

$$|\eta\rangle := (T \otimes T^2)\Big(|+\rangle \otimes |+\rangle\Big) = \frac{1}{2}\Big(|0\rangle + \omega|1\rangle\Big) \otimes \Big(|0\rangle + \bar{\omega}|1\rangle\Big).$$

Next, applying the gate U_+ to $|\eta\rangle$ gives

$$U_+|\eta\rangle = \frac{1}{2}\Big(|00\rangle + \bar{\omega}|01\rangle + \omega|11\rangle + |12\rangle\Big).$$

Now compare the first qubit with the state $|\tilde{0}\rangle$ and suppose we get "yes". Then, resulting (unnormalized) state is

$$\big(\langle\tilde{0}| \otimes \mathbb{1}\big)U_+|\eta\rangle = \frac{1}{2\sqrt{3}}\Big(|0\rangle + \bar{\omega}|1\rangle + \omega|1\rangle + |2\rangle\Big) = \frac{1}{2\sqrt{3}}\Big(|0\rangle - |1\rangle + |2\rangle\Big)$$

which is exactly the state $|\xi\rangle$ (after normalizing). Thus we see that we can construct the state $|\xi\rangle$ with success probability 1/4.

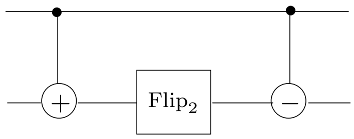
(c) (*Extra credit*) Given $|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$, apply U_+ to $|\psi\rangle \otimes |\xi\rangle$:

$$U_+\Big(|\psi\rangle \otimes |\xi\rangle\Big) = \frac{1}{\sqrt{3}}\Big[c_0|0\rangle \otimes \Big(|0\rangle - |1\rangle + |2\rangle\Big) + c_1|1\rangle \otimes \Big(|1\rangle - |2\rangle + |0\rangle\Big) + c_2|2\rangle \otimes \Big(|2\rangle - |0\rangle + |1\rangle\Big)\Big].$$

Now measure the second qubit in the standard basis. We get oucomes 0, 1 or 2 with equal probability. If we get 0, the state of the first qubit is $|\psi'\rangle = c_0|0\rangle + c_1|1\rangle - c_2|2\rangle$, i.e. we flipped the sign of c_2; if we get 1, $|\psi'\rangle = -c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$, i.e. we flipped c_0; and if we get 2, $|\psi'\rangle = c_0|0\rangle - c_1|1\rangle + c_2|2\rangle$, i.e. we flipped c_1. If we did not flip the coefficient we wanted, we can repeat the procedure (on the same state). We can figure out when we are successful by keeping track of which sign we flipped each time.

(d) (*Extra credit*) Let us work with qubit input states only, so the action of $\Lambda^2(-1)$ is to flip the sign of the coefficient of $|11\rangle$. The idea behind implementing this gate is simple: use the U_+ gate to get the state $|2\rangle$ on one of the two qutrits, which will only occur if the two qutrits were originally in state $|11\rangle$, apply the sign flip to the state $|2\rangle$, and then undo the U_+ gate using the U_- gate. The following circuit does precisely this:

Figure 2: $\Lambda^2(-1)$ gate.



where the gate Flip_2 refers to the sign flip gate (from part (c)) which flips the sign of the coefficient for state $|2\rangle$.

For $\Lambda^3(-1)(\sigma_z)^{\otimes 3}$, the gate flips the sign of the coefficients of $|001\rangle$, $|010\rangle$ and $|100\rangle$. This gate can be implemented using a similar idea as the $\Lambda^2(-1)$ gate:

Figure 3: $\Lambda^3(-1)(\sigma_z)^{\otimes 3}$ gate.