Lecture Notes for Physics 219: Quantum Computation

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Topological quantum computation

9.1 Anyons, anyone?

A central theme of quantum theory is the concept of *indistinguishable particles* (also called *identical particles*). For example, all electrons in the world are exactly alike. Therefore, for a system with many electrons, an operation that *exchanges* two of the electrons (swaps their positions) is a symmetry — it leaves the physics unchanged. This symmetry is represented by a unitary transformation acting on the many-electron wave function.

For the indistinguishable particles in three-dimensional space that we normally talk about in physics, particle exchanges are represented in one of two distinct ways. If the particles are bosons (like, for example, ^{4}He atoms in a superfluid), then an exchange of two particles is represented by the identity operator: the wave function is invariant, and we say the particles obey Bose statistics. If the particles are fermions (like, for example, electrons in a metal), than an exchange is represented by multiplication by (-1): the wave function changes sign, and we say that the particles obey Fermi statistics.

The concept of identical-particle statistics becomes ambiguous in one spatial dimension. The reason is that for two particles to swap positions in one dimension, the particles need to pass through one another. If the wave function changes sign when two identical particles are exchanged, we could say that the particles are noninteracting fermions, but we could just as well say that the particles are interacting bosons, such that the sign change is induced by the interaction as the particles pass one another. More generally, the exchange could modify the wavefunction by a multiplicative phase $e^{i\theta}$ that could take values other than $+1$ or -1, but we could account for this phase change by describing the particles as either bosons or fermions.

Thus, identical-particle statistics is a rather tame concept in three (or more) spatial dimensions and also in one dimension. But in between these two dull cases, in two dimensions, a remarkably rich variety of types of particle statistics are possible, so rich that we have far to go before we can give a useful classification of all of the possibilities.

Indistinguishable particles in two dimensions that are neither bosons nor fermions are called *anyons*. Anyons are a fascinating theoretical construct, but do they have anything to do with the physics of real systems that can be studied in the laboratory? The remarkable answer is: "Yes!" Even in our three-dimensional world, a two-dimensional gas of electrons can be realized by trapping the electrons in a thin layer between two slabs of semiconductor, so that at low energies, electron motion in the direction orthogonal to the layer is frozen out. In a sufficiently strong magnetic field and at sufficiently low temperature, and if the electrons in the material are sufficiently mobile, the two-dimensional electron gas attains a profoundly entangled ground state that is separated from all excited states by a nonzero energy gap. Furthermore, the low-energy particle excitations in the systems do not have the quantum numbers of electrons; rather they are anyons, and carry electric charges that are fractions of the electron charge. The anyons have spectacular effects on the transport properties of the sample, manifested as the fractional quantum Hall effect.

Anyons will be our next topic. But why? True, I have already said enough to justify that anyons are a deep and fascinating subject. But this is not a course about the unusual behavior of exotic phases attainable in condensed matter systems. It is a course about quantum computation.

In fact, there is a connection, first appreciated by Alexei Kitaev in 1997: anyons provide an unusual, exciting and perhaps promising means of realizing fault-tolerant quantum computation.

So that sounds like something we should be interested in. After all, I have already given 12 lectures on the theory of quantum error correction and fault-tolerant quantum computing. It is a beautiful theory; I have enjoyed telling you about it and I hope you enjoyed hearing about it. But it is also daunting. We've seen that an ideal quantum circuit can be simulated faithfully by a circuit with noisy gates, provided the noisy gates are not *too* noisy, and we've seen that the overhead in circuit size and depth required for the simulation to succeed is reasonable. These observations greatly boost our confidence that large scale quantum computers will really be built and operated someday. Still, for fault tolerance to be effective, quantum gates need to have quite high fidelity (by the current standards of experimental physics), and the overhead cost of achieving fault tolerance is substantial. Even though reliable quantum computation with noisy gates is possible in principle, there always will be a strong incentive to improve the fidelity of our computation by improving the hardware rather than by compensating for the deficiencies of the hardware through clever circuit design. By using anyons, we might achieve fault tolerance by designing hardware with an intrinsic resistance to decoherence and other errors, significantly reducing the size and depth blowups of our circuit simulations. Clearly, then, we have ample motivation for learning about anyons. Besides, it will be fun!

In some circles, this subject has a reputation (not fully deserved in my view) for being abstruse and inaccessible. I intend to start with the basics, and not to clutter the discussion with details that are mainly irrelevant to our central goals. That way, I hope to keep the presentation clear without really dumbing it down.

What are these goals? I will *not* be explaining how the theory of anyons connects with observed phenomena in fractional quantum Hall systems. In particular, *abelian* anyons arise in most of these applications. From a quantum information viewpoint, abelian anyons are relevant to robust *storage* of quantum information (and we have already gotten a whiff of that connection in our study of toric quantum codes). We will discuss abelian anyons here, but our main interest will be in *nonabelian* anyons, which as we will see can be endowed with surprising computational power.

Kitaev (quant-ph/9707021) pointed out that a system of nonabelian anyons with suitable properties can efficiently simulate a quantum circuit; this idea was elaborated by Ogburn and me (quant-ph/9712048), and generalized by Mochon (quant-ph/0206128, quant-ph/0306063). In Kitaev's original scheme, measurements were required to simulate some quantum gates. Freedman, Larsen and Wang (quant-ph/000110) observed that if we use the right kind of anyons, all measurements can be postponed until the readout of the final result of the computation. Freedman, Kitaev, and Wang (quant-ph/0001071) also showed that a system of anyons can be simulated efficiently by a quantum circuit; thus the anyon quantum computer and the quantum circuit model have equivalent computational power. The aim of these lectures is to explain these important results.

We will focus on the applications of anyons to quantum computing, not on the equally important issue of how systems of anyons with desirable properties can be realized in practice.* It will be left to you to figure that out!

* Two interesting approaches to realizing nonabelian anyons — using superconducting junction arrays and using cold atoms trapped in optical lattices — have been discussed in the recent literature.

9.2 Flux-charge composites

For those of us who are put off by abstract mathematical constructions, it will be helpful to begin our exploration of the theory of anyons by thinking about a concrete model. So let's start by recalling a more familiar concept, the *Aharonov-Bohm effect*.

Imagine electromagnetism in a two-dimensional world, where a "flux tube" is a localized "pointlike" object (in three dimensions, you may envision a plane intersecting a magnetic solenoid directed perpendicular to the plane). The flux might be enclosed behind an impenetrable wall, so that an object outside can never visit the region where the magnetic field is nonzero. But even so, the magnetic field has a measurable influence on charged particles outside the flux tube. If an electric charge q is adiabatically transported (counterclockwise) around a flux Φ, the wave function of the charge acquires a *topological phase* $e^{iq\Phi}$ (where we use units with $\hbar = c = 1$). Here the world "topological" means that the Aharonov-Bohm phase is robust when we deform the trajectory of the charged particle — all that matters is the "winding number" of the charge about the flux.

The concept of topological invariance arises naturally in the study of fault tolerance. Topological properties are those that remain invariant when we smoothly deform a system, and a fault-tolerant quantum gate is one whose action on protected information remains invariant (or nearly so) when we deform the implementation of the gate by adding noise. The topological invariance of the Aharonov-Bohm phenomenon is the essential property that we hope to exploit in the design of quantum gates that are intrinsically robust.

We usually regard the Aharonov-Bohm effect as a phenomenon that occurs in quantum electrodynamics, where the photon is exactly massless. But it is useful to recognize that Aharonov-Bohm phenomena can also occur in massive theories. For example, we might consider a "superconducting" system composed of charge e particles, such that composite objects with charge ne form a condensate (where n is an integer). In this superconductor, there is a quantum of flux $\Phi_0 = 2\pi/ne$, the minimal nonzero flux such that a charge-(ne) particle in the condensate, when transported around the flux, acquires a *trivial* Aharonov-Bohm phase. An isolated region that contains a flux quantum is an island of normal material surrounded by the superconducting condensate, prevented from spreading because the magnetic flux cannot penetrate into the superconductor. That is, it is a stable particle, what we could call a "fluxon." When one of the charge-e particles is transported around a fluxon, its wave function acquires the nontrivial topological phase $e^{ie\Phi_0} = e^{2\pi i/n}$. But in the superconductor, the photon acquires a mass via the Higgs mechanism, and there are no massless particles. That topological phases are compatible with massive theories is important, because massless particles are easily excited, a potentially copious source of decoherence.

Now, let's imagine that, in our two-dimensional world, flux and electric charge are permanently bound together (for some reason). A fluxon can be envisioned as flux Φ confined inside an impenetrable circular wall, and an electric charge q is stuck to the *outside* of the wall. What is the angular momentum of this flux-charge composite? Suppose that we carefully rotate the object counterclockwise by angle 2π, returning it to its original orientation. In doing so, we have transported the charge q about the flux Φ, generating a topological phase $e^{iq\Phi}$. This rotation by 2π is represented in Hilbert space by the unitary transformation

$$\boldsymbol{U}(2\pi) = e^{-i2\pi\boldsymbol{J}} = e^{iq\Phi} \ , \tag{9.1}$$

where $\boldsymbol{J}$ is the angular momentum. We conclude, then, that the possible eigenvalues of angular momentum are

$$J = m - \frac{q\Phi}{2\pi} \quad (m = \text{integer}) \ . \tag{9.2}$$

We can characterize this spectrum by an angular variable $\theta \in [0, 2\pi)$, defined by $\theta = q\Phi \pmod{2\pi}$, and say that the eigenvalues are shifted away from integer values by $-\theta/2\pi$. We will refer to the phase $e^{i\theta}$ that represents a counterclockwise rotation by 2π as the *topological spin* of the composite object.

But shouldn't a rotation by 2π act trivially on a physical system (isn't it the same as doing nothing)? No, we know better than that, from our experience with *spinors* in three dimensions. For a system with fermion number F, we have

$$e^{-2\pi i J} = (-1)^F \ ; \tag{9.3}$$

if the fermion number is odd, the eigenvalues of $\boldsymbol{J}$ are shifted by $1/2$ from the integers. This shift is physically acceptable because there is a $(-1)^F$ *superselection rule*: no observable local operator can change the value of $(-1)^F$ (there is no physical process that can create or destroy an isolated fermion). Acting on a coherent superposition of states with different values of $(-1)^F$, the effect of $e^{-2\pi i\boldsymbol{J}}$ is

$$e^{-i2\pi\boldsymbol{J}} \left(a|\text{ even } F\ \rangle + b|\text{ odd } F\ \rangle\right) = a|\text{ even } F\ \rangle - b|\text{ odd } F\ \rangle \ . \tag{9.4}$$

The relative sign in the superposition flips, but this has no detectable physical effects, since all observables are block diagonal in the $(-1)^F$ basis.

Similarly, in two dimensions, the shift in the angular momentum spectrum $e^{-2\pi i\boldsymbol{J}} = e^{i\theta}$ has no unacceptable physical consequences if there is a θ superselection rule, ensuring that the relative phase in a superposition of states with different values of θ is physically inaccessible (not just in practice but even in principle). As for fermions, there is no allowed physical process that can create of destroy an isolated anyon.

In three dimensions, only $\theta = 0, \pi$ are allowed, because (as you probably know) of a topological property of the three-dimensional rotation group SO(3): a closed path in SO(3) beginning at the identity and ending at a rotation by 4π can be smoothly contracted to a trivial path. It follows that a rotation by 4π really is represented by the identity, and therefore that the eigenvalues of a rotation by 2π are $+1$ and -1. But the two-dimensional rotation group SO(2) does not have this topological property, so that any value of θ is possible in principle.

Note that the angular momentum $\boldsymbol{J}$ changes sign under time reversal ($\boldsymbol{T}$) and also under parity ($\boldsymbol{P}$). Unless $\theta = 0$ or π, the spectrum of $\boldsymbol{J}$ is asymmetric about zero, and therefore a theory of anyons typically will not be $\boldsymbol{T}$ or $\boldsymbol{P}$ invariant. In our flux-charge composite model the origin of this symmetry breaking is not mysterious — it arises from the nonzero magnetic field. But in a system with no intrinsic breaking of $\boldsymbol{T}$ and $\boldsymbol{P}$, if anyons occur then either these symmetries must be broken spontaneously, or else the particle spectrum must be "doubled" so that for each anyon with exchange phase $e^{i\theta}$ there also exists an otherwise identical particle with exchange phase $e^{-i\theta}$.

9.3 Spin and statistics

For identical particles in three dimensions, there is a well known connection between spin and statistics: indistinguishable particles with integer spin are bosons, and those with half-odd-integer spin are fermions. In two dimensions, the spin can be any real number. What does this new possibility of "fractional spin" imply about statistics? The answer is that statistics, too, can be "fractionalized"!

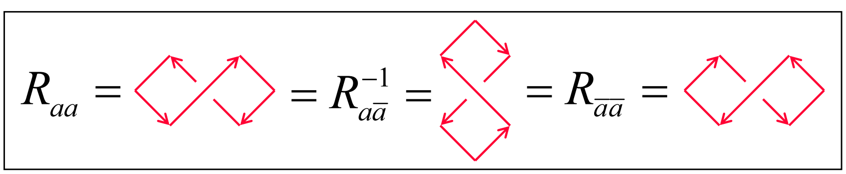
What happens if we perform an exchange of two of our flux-charge composite objects, in a counterclockwise sense? Each charge q is adiabatically transported *half way* around the flux Φ of the other object. We can anticipate, then, that each charge will acquire an Aharonov-Bohm phase that is half of the phase generated by a complete revolution of the charge about the flux. Adding together the phases arising from the transport of both charges, we find that the exchange of the two flux-charge composites changes their wave function by the phase

$$\exp\left[i\left(\frac{1}{2}q\Phi + \frac{1}{2}q\Phi\right)\right] = e^{iq\Phi} = e^{i\theta} = e^{-2\pi i\boldsymbol{J}} \ . \tag{9.5}$$

The phase generated when the two objects are exchanged matches the phase generated when one of the two objects is rotated by 2π. Thus the connection between spin and statistics continues to hold, in a form that is a natural generalization of the connection that applies to bosons and fermions.

The origin of this connection is fairly clear in our flux-charge composite model, but in fact it holds much more generally. Why? Reading textbooks on relativistic quantum field theory, one can easily get the impression that the spin-statistics connection is founded on Lorentz invariance, and has something to do with the properties of the complexified Lorentz group. Actually, this impression is quite misleading. All that is essential for a spin-statistics connection to hold is *the existence of antiparticles*. Special relativity is not an essential ingredient.

Consider an anyon, characterized by the phase θ, and suppose that this particle has a corresponding antiparticle. This means that the particle and its antiparticle, when combined, have trivial quantum numbers (in particular, zero angular momentum) and therefore that there are physical processes in which particle-antiparticle pairs can be created and annihilated. Draw a world line in spacetime that represents a process in which two particle-antiparticle pairs are created (one pair on the left and the other pair on the right), the particle from the pair on the right is exchanged *in a counterclockwise sense* with the particle from the pair on the left, and then both pairs reannihilate. (The world line has an orientation; if directed forward in time it represents a particle, and if directed backward in time it represents an antiparticle.) Turning our diagram 90°, we obtain a depiction of a process in which a single particle-antiparticle pair is created, the particle and antiparticle are exchanged in a *clockwise* sense, and then the pair reannihilates. Turning it 90° yet again, we have a process in which two pairs are created and the *antiparticle* from the pair on the right is exchanged, in a counterclockwise sense, with the antiparticle from the pair on the left, before reannihilation.



What do we conclude from these manipulations? Denote by R_{ab} the unitary operator that represents a counterclockwise exchange of particles of types a and b (so that the inverse operator R^{-1}_{ab} represents a clockwise exchange), and denote by $\bar{a}$ the antiparticle of a. We have found that

$$R_{aa} = R^{-1}_{a\bar{a}} = R_{\bar{a}\bar{a}} \ . \tag{9.6}$$

If a is an anyon with exchange phase $e^{i\theta}$, then its antiparticle $\bar{a}$ also has the *same* exchange phase. Furthermore, when a and $\bar{a}$ are exchanged counterclockwise, the phase acquired is $e^{-i\theta}$.

These conclusions are unsurprising when we interpret them from the perspective of our flux-charge composite model of anyons. The antiparticle of the object with flux Φ and charge q has flux $-\Phi$ and charge $-q$. Hence, when we exchange two antiparticles, the minus signs cancel and the effect is the same as though the particles were exchanged. But if we exchange a particle and an antiparticle, then the relative sign of charge and flux results in the exchange phase $e^{-iq\Phi} = e^{-i\theta}$.

But what is the connection between these observations about statistics and the spin? Continuing to contemplate the same spacetime diagram, let us consider its implications regarding the *orientation* of the particles. For keeping track of the orientation, it is convenient to envision the particle world line not as a thread but as a *ribbon* in spacetime. I claim that our process can be smoothly deformed to one in which a particle-antiparticle pair is created, the particle is rotated counterclockwise by 2π, and then the pair reannihilates. A convenient way to verify this assertion is to take off your belt (or borrow a friend's). The buckle at one end specifies an orientation; point your thumb toward the buckle, and following the right-hand rule, twist the belt by 2π before rebuckling it. You should be able to check that you can lay out the belt to match the spacetime diagram for any of the exchange processes described earlier, and also for the process in which the particle rotates by 2π.

Thus, in a topological sense, rotating a particle counterclockwise by 2π is really the same thing as exchanging two particles in a counterclockwise sense (or exchanging particle and antiparticle in a clockwise sense), which provides a satisfying explanation for a general spin-statistics connection.† I emphasize again that this argument invokes processes in which particle-antiparticle pairs are created and annihilated, and therefore the existence of antiparticles is an essential prerequisite for it to apply.

9.4 Combining anyons

We know that a composite object composed of two fermions is a boson. What happens when we build a composite object by combining two anyons?

† Actually, this discussion has been oversimplified. Though it is adequate for abelian anyons, we will see that it must be amended for nonabelian anyons, because R_{ab} has more than one eigenvalue in the nonabelian case. Similarly, the discussion in the next section of "combining anyons" will need to be elaborated because, in the nonabelian case, more than one kind of composite anyon can be obtained when two anyons are fused together.

Suppose that a is an anyon with exchange phase $e^{i\theta}$, and that we build a "molecule" from n of these a anyons. What phase is acquired under a counterclockwise exchange of the two molecules?

The answer is clear in our flux-charge composite model. Each of the n charges in one molecule acquires a phase $e^{i\theta/2}$ when transported half way around each of the n fluxes in the other molecule. Altogether then, $2n^2$ factors of the phase $e^{i\theta/2}$ are generated, resulting in the total phase

$$e^{i\theta_n} = e^{in^2\theta} \ . \tag{9.7}$$

Said another way, the phase $e^{i\theta}$ occurs altogether n^2 times because in effect n anyons in one molecule are being exchanged with n anyons in the other molecule. Contrary to what we might have naively expected, if we split a fermion (say) into two identical constituents, the constituents have, not an exchange phase of $\sqrt{-1} = i$, but rather $(e^{i\pi})^{1/4} = e^{i\pi/4}$.

This behavior is compatible with the spin-statistics connection: the angular momentum J of the n-anyon molecule satisfies

$$e^{-2\pi i J_n} = e^{-2\pi i n^2 J} = e^{in^2\theta} \ . \tag{9.8}$$

For example, consider a molecule of two anyons, and imagine rotating the molecule counterclockwise by 2π. Not only does each anyon in the molecule rotate by 2π; in addition one of the anyons revolves around the other. One revolution is equivalent to two successive exchanges, so that the phase generated by the revolution is $e^{i2\theta}$. The total effect of the two rotations and the revolution is the phase

$$\exp\left[i\left(\theta + \theta + 2\theta\right)\right] = e^{i4\theta} \ . \tag{9.9}$$

Another way to understand why the angular momenta of the anyons in the molecule do not combine additively is to note that the total angular momentum of the molecule consists of two parts — the spin angular momentum S of each of the two anyons (which *is* additive) and the *orbital* angular momentum L of the anyon pair. Because the counterclockwise transport of one anyon around the other generates the nontrivial phase $e^{i2\theta}$, the dependence of the two-anyon wavefunction ψ on the relative azimuthal angle φ is not single-valued; instead,

$$\psi(\varphi + 2\pi) = e^{-i2\theta}\psi(\varphi) \ . \tag{9.10}$$

This means that the spectrum of the orbital angular momentum $\boldsymbol{L}$ is shifted away from integer values:

$$e^{-i2\pi \boldsymbol{L}} = e^{2i\theta} \ , \tag{9.11}$$

and this orbital angular momentum combines additively with the spin S to produce the total angular momentum

$$-2\pi J = -2\pi L - 2\pi S = 2\theta + 2\theta + 2\pi(\text{integer}) = 4\theta + 2\pi(\text{integer})\ . \quad (9.12)$$

What if, on the other hand, we build a molecule $\bar{a}a$ from an anyon a and its antiparticle $\bar{a}$? Then, as we've seen, the spin S has the same value as for the aa molecule. But the exchange phase has the opposite value, so that the noninteger part of the orbital angular momentum is $-2\pi L = -2\theta$ instead of $-2\pi L = 2\theta$, and the total angular momentum $J = L + S$ is an integer. This property is necessary, of course, if the $\bar{a}a$ pair is to be able to annihilate without leaving behind an object that carries nontrivial angular momentum.

9.5 Unitary representations of the braid group

We have already noted that the angular momentum spectrum has different properties in two spatial dimensions than in three dimensions because SO(2) has different topological properties than SO(3) (SO(3) has a compact simply connected covering group SU(2), but SO(2) does not). This observation provides one way to see why anyons are possible in two dimensions but not in three. It is also instructive to observe that particle *exchanges* have different topological properties in two spatial dimensions than in three dimensions.

As we have found in our discussion of the relation between the statistics of particles and of antiparticles, it is useful to envision exchanges of particles as processes taking place in spacetime. In particular, it is convenient to imagine that we are computing the quantum transition amplitude for a time-dependent process involving n particles by evaluating a sum over particle histories (though for our purposes it will not actually be necessary to calculate any path integrals).

Consider a system of n indistinguishable pointlike particles confined to a two-dimensional spatial surface (which for now we may assume is the plane), and suppose that no two particles are permitted to occupy coincident positions. We may think of a configuration of the particles at a fixed time as a plane with n "punctures" at specified locations — that is, we associate with each particle a hole in the surface with infinitesimal radius. The condition that the particles are forbidden to coincide is enforced by demanding that there are exactly n punctures in the plane at any time. Furthermore, just as the particles are indistinguishable, each puncture is the same as any other. Thus if we were to perform a permutation of the n punctures, this would have no physical effect; all the punctures are the same anyway, so it makes no difference which one is which. All that matters is the n distinct particle positions in the plane.

To evaluate the quantum amplitude for a configuration of n particles at specified initial positions at time $t = 0$ to evolve to a configuration of n particles at specified final positions at time $t = T$, we are to sum over all classical histories for the n particles that interpolate between the fixed initial configuration and the fixed final configuration, weighted by the phase e^{iS}, where S is the classical action of the history. If we envision each particle world line as a thread, each history for the n particles becomes a *braid*, where each particle on the initial ($t = 0$) time slice can be connected by a thread to any one of the particles on the final ($t = T$) time slice. Furthermore, since the particle world lines are forbidden to cross, the braids fall into distinct topological classes that cannot be smoothly deformed one to another, and the path integral can be decomposed as a sum of contributions, with each contribution arising from a different topological class of histories.

Nontrivial exchange operations acting on the particles on the final time slice change the topological class of the braid. Thus we see that the elements of the symmetry group generated by exchanges are in one-to-one correspondence with the topological classes. This (infinite) group is called B_n, the braid group on n strands; the group composition law corresponds to concatenation of braids (that is, following one braid with another). In the quantum theory, the quantum state of the n indistinguishable particles belongs to a Hilbert space that transforms as a unitary representation of the braid group B_n.

The group can be presented as a set of generators that obey particular defining relations. To understand the defining relations, we may imagine that the n particles occupy n ordered positions (labeled $1, 2, 3, \ldots, n$) arranged on a line. Let σ_1 denote a counterclockwise exchange of the particles that initially occupy positions 1 and 2, let σ_2 denote a counterclockwise exchange of the particles that initially occupy positions 2 and 3, and so on. Any braid can be constructed as a succession of exchanges of neighboring particles; hence $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are the group generators.

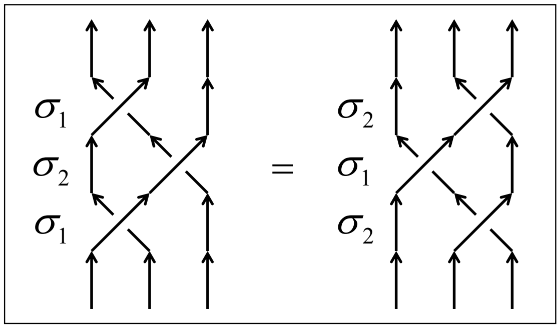
The defining relations satisfied by these generators are of two types. The first type is

$$\sigma_j \sigma_k = \sigma_k \sigma_j \ , \quad |j - k| \geq 2 \ , \tag{9.13}$$

which just says that exchanges of disjoint pairs of particles commute. The second, slightly more subtle, type of relation is

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \ , \quad j = 1, 2, \ldots, n-2 \ , \tag{9.14}$$

which is sometimes called the *Yang-Baxter relation.* You can verify the Yang-Baxter relation by drawing the two braids $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ on a piece of paper, and observing that both describe a process in which the particles initially in positions 1 and 3 are exchanged counterclockwise about the particle labeled 2, which stays fixed — *i.e.*, these are topologically equivalent braids.



Since the braid group is infinite, it has an infinite number of unitary irreducible representations, and in fact there are an infinite number of *one-dimensional* representations. Indistinguishable particles that transform as a one-dimensional representation of the braid group are said to be *abelian anyons*. In the one-dimensional representations, each generator σ_j of B_n is represented by a phase $\sigma_j = e^{i\theta_j}$. Furthermore, the Yang-Baxter relation becomes $e^{i\theta_j} e^{i\theta_{j+1}} e^{i\theta_j} = e^{i\theta_{j+1}} e^{i\theta_j} e^{i\theta_{j+1}}$, which implies $e^{i\theta_j} = e^{i\theta_{j+1}} \equiv e^{i\theta}$ — all exchanges are represented by the *same* phase. Of course, that makes sense; if the particles are really indistinguishable, the exchange phase ought not to depend on which pair is exchanged. For $\theta = 0$ we obtain bosons, and for $\theta = \pi$, fermions

The braid group also has many nonabelian representations that are of dimension greater than one; indistinguishable particles that transform as such representations are said to be *nonabelian anyons* (or, sometimes, *nonabelions*). To understand the physical properties of nonabelian anyons we will need to understand the mathematical structure of some of these representations. In these lectures, I hope to convey some intuition about nonabelian anyons by discussing some examples in detail.

For now, though, we can already anticipate the main goal we hope to fulfill. For nonabelian anyons, the irreducible representation of B_n realized by n anyons acts on a "topological vector space" V_n whose dimension D_n increases exponentially with n. And for anyons with suitable properties, the image of the representation may be *dense* in $\mathrm{SU}(D_n)$. Then braiding of anyons can simulate a quantum computation — any (special) unitary transformation acting on the exponentially large vector space V_n can be realized with arbitrarily good fidelity by executing a suitably chosen braid.

Thus we are keenly interested in the nonabelian representations of the braid group. But we should also emphasize (and will discuss at greater length later on) that there is more to a model of anyons than a mere representation of the braid group. In our flux tube model of abelian anyons, we were able to describe not only the effects of an exchange of anyons, but also the types of particles that can be obtained when two or more anyons are combined together. Likewise, in a general anyon model, the anyons are of various types, and the model incorporates "fusion rules" that specify what types can be obtained when two anyons of particular types are combined. Nontrivial consistency conditions arise because fusion is associate (fusing a with b and then fusing the result with c is equivalent to fusing b with c and then fusing the result with a), and because the fusion rules must be consistent with the braiding rules. Though these consistency conditions are highly restrictive, many solutions exist, and hence many different models of nonabelian anyons are realizable in principle.

9.6 Topological degeneracy

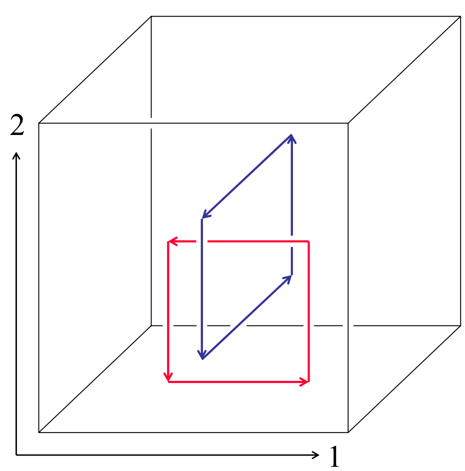
But before moving on to nonabelian anyons, there is another important idea concerning abelian anyons that we should discuss. In any model of anyons (indeed, in any local quantum system with a mass gap), there is a ground state or *vacuum state*, the state in which no particles are present. On the plane the ground state is unique, but for a two-dimensional surface with nontrivial topology, the ground state is degenerate, with the degree of degeneracy depending on the topology. We have already encountered this phenomenon of "topological degeneracy" in the model of abelian anyons that arose in our study of a particular quantum error-correcting code, Kitaev's toric code. Now we will observe that topological degeneracy is a general feature of any model of (abelian) anyons.

We can arrive at the concept of topological degeneracy by examining the representations of a simple operator algebra. Consider the case of the torus, represented as a square with opposite sides identified, and consider the two fundamental 1-cycles of the torus: C_1, which winds around the square in the x_1 direction, and C_2 which winds around in the x_2 direction. A unitary operator T_1 can be constructed that describes a process in which an anyon-antianyon pair is created, the anyon propagates around C_1, and then the pair reannihilates. Similarly a unitary operator T_2 can be constructed that describes a process in which the pair is created, and the anyon propagates around the cycle C_2 before the pair reannihilates. Each of the operators T_1 and T_2 preserves the ground state of the system (the state with no particles); indeed, each commutes with the Hamiltonian H of the system and so either can be simultaneously diagonalized with H (T_1 and T_2 are both symmetries).

However, T_1 and T_2 do not commute with one another. If our torus has infinite spatial volume, and there is a mass gap (so that the only interactions among distantly separated anyons are due to the Aharonov-Bohm effect), then the commutator of T_1 and T_2 is

$$T_2^{-1}T_1^{-1}T_2T_1 = e^{-i2\theta}I \ , \tag{9.15}$$

where $e^{i\theta}$ is the anyon's exchange phase. The nontrivial commutator arises because the process in which (1) an anyon winds around C_1, (2) an anyon winds around C_2 (3) an anyon winds around C_1 in the reverse direction, and (4) an anyon winds around C_2 in the reverse direction, is topologically equivalent to a process in which one anyon winds clockwise around another. To verify this claim, view the action of $T_2^{-1}T_1^{-1}T_2T_1$ as a process in spacetime. First note that the process described by the operator $T_1^{-1}T_1$, in which an anyon world line first sweeps though C_1 and then immediately traverses C_1 in the reverse order, can be deformed to a process in which the anyon world line traverses a topologically trivial loop that can be smoothly shrunk to a point (in keeping with the property that $T_1^{-1}T_1$ is really the identity operator). In similar fashion, the process described by the operator $T_2^{-1}T_1^{-1}T_2T_1$ can be deformed to one where the anyon world lines traverse two closed loops, but such that the world lines *link* once with one another; furthermore, one loop pierces the surface bounded by the other loop in a direction opposite to the orientation inherited by the surface via the right-hand rule from its bounding loop. This process can be smoothly deformed to one in which two pairs are created, one anyon winds clockwise around the other, and then both pairs annihilate. The clockwise winding is equivalent to two successive clockwise exchanges, represented in our one-dimensional representation of the braid group by the phase $e^{-i2\theta}$. We conclude that T_1 and T_2 are noncommuting, except in the cases $\theta = 0$ (bosons) and $\theta = \pi$ (fermions).



Since T_1 and T_2 both commute with the Hamiltonian H, both preserve the eigenspaces of H, but since T_1 and T_2 do not commute with one another, they cannot be simultaneously diagonalized. Since T_1 is unitary, its eigenvalues are phases; let us use the angular variable $\alpha \in [0, 2\pi)$ to label an eigenstate of T_1 with eigenvalue $e^{i\alpha}$:

$$T_1|\alpha\rangle = e^{i\alpha}|\alpha\rangle \ . \tag{9.16}$$

Then applying T_2 to the T_1 eigenstate advances the value of α by 2θ:

$$T_1 \left(T_2|\alpha\rangle\right) = e^{i2\theta} T_2 T_1|\alpha\rangle = e^{i2\theta} e^{i\alpha} \left(T_2|\alpha\rangle\right) \ . \tag{9.17}$$

Suppose that θ is a *rational* multiple of 2π, which we may express as

$$\theta = \pi p/q \ , \tag{9.18}$$

where q and p ($p < 2q$) are positive integers with no common factor. Then we conclude that T_1 must have at least q distinct eigenvalues; T_1 acting on α generates an orbit with q distinct values:

$$\alpha + \left(\frac{2\pi p}{q}\right) k \ (\mathrm{mod}\ 2\pi) \ , \quad k = 0, 1, 2, \ldots, q-1 \ . \tag{9.19}$$

Since T_1 commutes with H, on the torus the ground state of our anyonic system (indeed, any energy eigenstate) must have a degeneracy that is an integer multiple of q. Indeed, generically (barring further symmetries or accidental degeneracies), the degeneracy is expected to be exactly q.

For a two-dimensional surface with genus g (a sphere with g "handles"), the degree of this topological degeneracy becomes q^g, because there are operators analogous to T_1 and T_2 associated with each of the g handles, and all of the T_1-like operators can be simultaneously diagonalized. Furthermore, we can apply a similar argument to a finite planar medium if *single* anyons can be created and destroyed at the *edges* of the system. For example, consider an annulus in which anyons can appear or disappear at the inner and outer edges. Then we could define the unitary operator T_1 as describing a process in which an anyon winds counterclockwise around the annulus, and a unitary operator T_2 as describing a process in which an anyon appears at the outer edge, propagates to the inner edge, and disappears. These operators T_1 and T_2 have the same commutator as the corresponding operators defined on the torus, and so we conclude as before that the ground state on the annulus is q-fold degenerate for $\theta = \pi p/q$. For a disc with h holes, there is an operator analogous to T_1 that winds an anyon counterclockwise around each of the holes, and an operator analogous to T_2 that propagates an anyon from the outer boundary of the disk to the edge of the hole; thus the degeneracy is q^h.

What we have described here is a robust *topological quantum memory.* The phase $e^{i2\theta} = e^{i2\pi p/q} \equiv \omega$ acquired when one anyon winds counterclockwise around another is a primitive qth root of unity, and in the case of a planar system with holes, the operator T_1 can be regarded as the encoded Pauli operator $\bar{Z}$ acting on a q-dimension system associated with a particular hole. Physically, the eigenvalue ω^s of $\bar{Z}$ just counts the number s of anyons that are "stuck" inside the hole. The operator T_2 can be regarded as the complementary Pauli operator $\bar{X}$ that increments the value of s by carrying one anyon from the boundary of the system and depositing it in the hole. Since the quantum information is encoded in a nonlocal property of the system, it is well protected from environmental decoherence. By the same token depositing a quantum state in the memory, and reading it out, might be challenging for this system, though in principle $\bar{Z}$ could be measured by, say, performing an interference experiment in which an anyon projectile scatters off of a hole. We will see later that by using nonabelian anyons we will be able to simplify the readout; in addition, with nonabelian anyons we can use topological properties to *process* quantum information as well as to store it.

Just how robust is this quantum memory? We need to worry about errors due to thermal fluctuations and due to quantum fluctuations. Thermal fluctuations might excite the creation of anyons, and thermal anyons might diffuse around one of the holes in the sample, or from one boundary to another, causing an encoded error. Thermal errors are heavily suppressed by the Boltzman factor $e^{-\Delta/T}$, if the temperature T is sufficiently small compared to the energy gap Δ (the minimal energy cost of creating a single anyon at the edge of the sample, or a pair of anyons in the bulk). The harmful quantum fluctuations are tunneling processes in which a virtual anyon-antianyon pair appears and the anyon propagates around a hole before reannihilating, or a virtual anyon appears at the edge of a hole and propagates to another boundary before disappearing. These errors due to quantum tunneling are heavily suppressed if the holes are sufficiently large and sufficiently well separated from one another and from the outer boundary.‡

Note that our conclusion that the topological degeneracy is finite hinged on the assumption that the angle θ is a rational multiple of π. We may say that a theory of anyons is *rational* if the topological degeneracy is finite for any surface of finite genus (and, for nonabelian anyons, if the

‡ If you are familiar with Euclidean path integral methods, you'll find it easy to verify that in the leading semiclassical approximation the amplitude A for such a tunneling process in which the anyon propagates a distance L has the form $A = Ce^{-L/L_0}$, where C is a constant and $L_0 = \hbar\,(2m^*\Delta)^{-1/2}$; here $\hbar$ is Planck's constant and m^* is the effective mass of the anyon, defined so that the kinetic energy of an anyon traveling at speed v is $\frac{1}{2}m^*v^2$.

topological vector space V_n is finite-dimensional for any finite number of anyons n). We may anticipate that the anyons that arise in any physically reasonable system will be rational in this sense, and therefore should be expected to have exchange phases that are roots of unity.

9.7 Toric code revisited

If these observations about topological degeneracy seem hauntingly familiar, it may be because we used quite similar arguments in our discussion of the toric code.

The toric code can be regarded as the (degenerate) ground state of a system of qubits that occupy the links of a square lattice on the torus, with Hamiltonian

$$H = -\frac{1}{4}\Delta \left(\sum_P Z_P + \sum_S X_S \right) , \tag{9.20}$$

where the plaquette operator $Z_P = \otimes_{\ell \in P} Z_\ell$ is the tensor product of Z's acting on the four qubits associated with the links contained in plaquette P, and the site operator $X_S \otimes_{\ell \ni S} X_\ell$ is the tensor product of X's acting on the four qubits associated with the links that meet at the site S. These plaquette and site operators are just the (commuting) stabilizer generators for the toric code. The ground state is the simultaneous eigenstate with eigenvalue 1 of all the stabilizer generators.

This model has two types of localized particle excitations — plaquette excitations where $Z_P = -1$, which we might think of as magnetic fluxons, and site excitations where $X_S = -1$, which we might think of as electric charges. A Z error acting on a link creates a pair of charges on the two site joined by the link, and an X error acting on a link creates a pair of fluxons on the two plaquettes that share the link. The energy gap Δ is the cost of creating a pair of either type.

The charges are bosons relative to one another (they have a trivial exchange phase $e^{i\theta} = 0$), and the fluxons are also bosons relative to one another. Since the fluxons are distinguishable from the charges, it does not make sense to exchange a charge with a flux. But what makes this an anyon model is that a phase (-1) is acquired when a charge is carried around a flux. The degeneracy of the ground state (the dimension of the code space) can be understood as a consequence of this property of the particles.

For this model on the torus, because there are two types of particles, there are two types of T_1 operators: $T_{1,S}$, which propagates a charge (site defect) around the 1-cycle C_1, and $T_{1,P}$, which propagates a fluxon (plaquette defect) around C_1. Similarly there are two types of T_2 operators,

$T_{2,S}$ and $T_{2,P}$. The nontrivial commutators are

$$T_{2,P}^{-1}T_{1,S}^{-1}T_{2,P}T_{1,S} = -1 = T_{2,S}^{-1}T_{1,P}^{-1}T_{2,S}T_{1,P} \ , \tag{9.21}$$

both arising from processes in which world lines of charges and fluxon link once with one another. Thus $T_{1,S}$ and $T_{2,S}$ can be diagonalized simultaneously, and can be regarded as the encoded Pauli operators $\bar{Z}_1$ and $\bar{Z}_2$ acting on two protected qubits. The operator $T_{2,P}$, which commutes with $\bar{Z}_1$ and anticommutes with $\bar{Z}_2$, can be regarded as the encoded $\bar{X}_1$, and similarly $T_{1,P}$ is the encoded $\bar{X}_2$.

On the torus, the degeneracy of the four ground states is exact for the ideal Hamiltonian we constructed (the particles have infinite effective masses). Weak local perturbations will break the degeneracy, but only by an amount that gets exponentially small as the linear size L of the torus increases. To be concrete, suppose the perturbation is a uniform "magnetic field" pointing in the $\hat{z}$ direction, coupling to the magnetic moments of the qubits:

$$H' = -h\sum_{\ell} Z_\ell \ . \tag{9.22}$$

Because of the nonzero energy gap, for the purpose of computing in perturbation theory the leading contribution to the splitting of the degeneracy, it suffices to consider the effect of the perturbation in the four-dimensional subspace spanned by the ground states of the unperturbed system. In the toric code, the operators with nontrivial matrix elements in this subspace are those such that Z_ℓ's act on links that form a closed loop that wraps around the torus (or X_ℓ's act on links whose dual links form a closed loop that wraps around the torus). For an $L \times L$ lattice on the torus, the minimal length of such a closed loop is L; therefore nonvanishing matrix elements do not arise in perturbation theory until the Lth order, and are suppressed by h^L. Thus, for small h and large L, memory errors due to quantum fluctuations occur only with exponentially small amplitude.

9.8 The nonabelian Aharonov-Bohm effect

There is a beautiful abstract theory of nonabelian anyons, and in due course we will delve into that theory a bit. But I would prefer to launch our study of the subject by describing a more concrete model.

With that goal in mind, let us recall some properties of *chromodynamics*, the theory of the quarks and gluons contained within atomic nuclei and other strongly interacting particles. In the real world, quarks are permanently bound together and can never be isolated, but for our discussion let us imagine a fictitious world in which the forces between quarks are weak, so that the characteristic distance scale of quark confinement is very large.

Quarks carry a degree of freedom known metaphorically as *color*. That is, there are three kinds of quarks, which in keeping with the metaphor we call red (R), yellow (Y), and blue (B). Quarks of all three colors are physical identical, except that when we bring two quarks together, we can tell whether their colors are the same (the Coulombic interaction between like colors is repulsive), or different (distinct colors attract). There is nothing to prevent me from establishing a *quark bureau of standards* in my laboratory, where colored quarks are sorted into three bins; all the quarks in the same bin have the same color, and quarks in different bins have different colors. We may attach (arbitrary) labels to the three bins — R, Y, and B.

If while taking a hike outside by lab, I discover a previously unseen quark, I may at first be unsure of its color. But I can find out. I capture the quark and carry it back to my lab, being very careful not to disturb its color along the way (in chromodynamics, there is a notion of *parallel transport* of color). Once back at the quark bureau of standards, I can compare this new quark to the previously calibrated quarks in the bins, and so determine whether the new quark should be labeled R, Y, or B.

It sounds simple but there is a catch: in chromodynamics, the parallel transport of color is *path dependent* due to an Aharonov-Bohm phenomenon that affects color. Suppose that at the quark bureau of standards a quark is prepared whose color is described by the quantum state

$$|\psi_q\rangle = q_R|R\rangle + q_Y|Y\rangle + q_B|B\rangle \ ; \tag{9.23}$$

it is a coherent superposition with amplitudes q_R, q_Y, q_B for the red, yellow, and blue states. The quark is carried along a path that winds around a *color magnetic flux tube* and is returned to the quark bureau of standards where its color can be recalibrated. Upon its return the color state has been rotated:

$$\begin{pmatrix} q'_R \\ q'_Y \\ q'_B \end{pmatrix} = U \begin{pmatrix} q_R \\ q_Y \\ q_B \end{pmatrix} \ , \tag{9.24}$$

where U is a (special) unitary 3×3 matrix. Similarly, when a newly discovered quark is carried back to the bureau of standards, the outcome of a measurement of its color will depend on whether it passed to the left or the right of the flux tube during its voyage.

This path dependence of the parallel transport of color is closely analogous to the path dependence of the parallel transport of a tangent vector on a curved Riemannian manifold. In chromodynamics, a magnetic field is the "curvature" whose strength determines the amount of path dependence.

In general, the SU(3) matrix U that describes the effect of parallel transport of color about a closed path depends on the *basepoint* x_0 where the path begins and ends, as well as on the closed loop C traversed by the path — when it is important to specify the loop and basepoint we will use the notation $U(C, x_0)$. The eigenvalues of the matrix U have an invariant "geometrical" meaning characterizing the parallel transport, but U itself depends on the conventions we have established at the basepoint. You might prefer to choose a different orthonormal basis for the color space at the basepoint x_0 than the basis I chose, so that your standard colors R, Y, and B differ from mine by the action of an SU(3) matrix $V(x_0)$. Then, while I characterize the effect or parallel transport around the loop C with the matrix U, you characterize it with another matrix

$$V(x_0)U(C, x_0)V(x_0)^{-1} \ , \tag{9.25}$$

that differs from mine by conjugation by $V(x_0)$. Physicists sometimes speak of this freedom to redefine conventions as a choice of *gauge*, and say that U itself is *gauge dependent* while its eigenvalues are *gauge invariant*.

Chromodynamics, on the distance scales we consider here (much smaller than the characteristic distance scale of quark confinement), is a theory like electrodynamics with long-range Coulombic interactions among quarks, mediated by "gluon" fields. We will prefer to consider a theory that retains some of the features of chromodynamics (in particular the path dependence of color transport), but without the easily excited light gluons. In the case of electrodynamics, we eliminated the light photon by considering a "superconductor" in which charged particles form a condensate, magnetic fields are expelled, and the magnetic flux of an isolated object is quantized. Let us appeal to the same idea here. We consider a *nonabelian superconductor* in two spatial dimensions. This world contains particles that carry "magnetic flux" (similar to the color magnetic flux in chromodynamics) and particles that carry charge (similar to the colored quarks of chromodynamics). The flux takes values in a *nonabelian finite group* G, and the charges are *unitary irreducible representations* of the group G. In this setting, we can formulate some interesting models of nonabelian anyons.

Let R denote a particular irreducible representation of G, whose dimension is denoted $|R|$. We may establish a "charge bureau of standards," and define there an arbitrarily chosen orthonormal basis for the $|R|$-dimensional vector space acted upon by R:

$$|R, i\rangle \ , \quad i = 1, 2, \ldots |R| \ . \tag{9.26}$$

When a charge R is transported around a closed path that encloses a flux $a \in G$, there is a nontrivial Aharonov-Bohm effect — the basis for R is rotated by a unitary matrix $D^R(a)$ that represents a:

$$|R,j\rangle \mapsto \sum_{i=1}^{|R|} |R,i\rangle D^R_{ij}(a) \ . \tag{9.27}$$

The matrix elements $D^R_{ij}(a)$ are measurable in principle, for example by conducting interference experiments in which a beam of calibrated charges can pass on either side of the flux. (The phase of the complex number $D^R_{ij}(a)$ determines the magnitude of the *shift* of the interference fringes, and the modulus of $D^R_{ij}(a)$ determines the *visibility* of the fringes.) Thus once we have chosen a standard basis for the charges, we can use the charges to attach labels (elements of G) to all fluxes. The flux labels are unambiguous as long as the representation R is faithful, and barring any group automorphisms (which create ambiguities that we are free to resolve however we please).

However, the group elements that we attach to the fluxes depend on our conventions. Suppose I am presented with k fluxons (particles that carry flux), and that I use my standard charges to measure the flux of each particle. I assign group elements $a_1, a_2, \ldots, a_k \in G$ to the k fluxons. You are then asked to measure the flux, to verify my assignments. But your standard charges differ from mine, because they have been surreptitiously transported around another flux (one that I would label with $g \in G$). Therefore you will assign the group elements $ga_1g^{-1}, ga_2g^{-1}, \ldots, ga_kg^{-1}$ to the k fluxons; our assignments differ by an overall conjugation by g.

The moral of this story is that the assignment of group elements to fluxons is inherently ambiguous and has no invariant meaning. But because the valid assignments of group elements to fluxons differ only by conjugation by some element $g \in G$, the *conjugacy class* of the flux in G does have an invariant meaning on which all observers will agree. Indeed, even if we fix our conventions at the charge bureau of standards, the group element that we assign to a particular fluxon may change if that fluxon takes part in a physical process in which *it* braids with other fluxons. For that reason, the fluxons belonging to the same conjugacy class should all be regarded as indistinguishable particles, even though they come in many varieties (one for each representative of the class) that can be distinguished when we make measurements at a particular time and place: The fluxons are *nonabelian anyons.*

9.9 Braiding of nonabelian fluxons

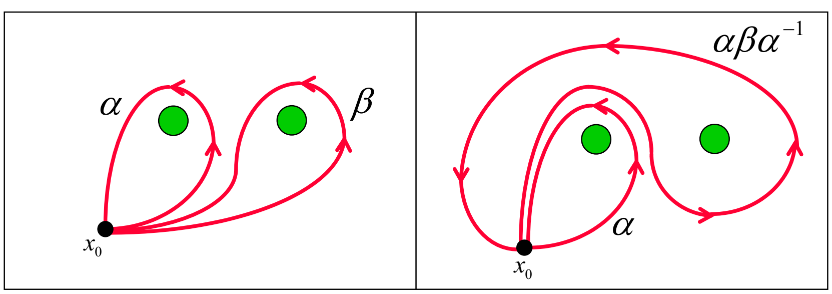
We will see that, for a nonabelian superconductor with suitable properties, it is possible to operate a fault-tolerant universal quantum computer by manipulating the fluxons. The key thing to understand is what happens when two fluxons are exchanged with one another.

For this purpose, imagine that we carefully calibrate two fluxons, and label them with elements of the group G. The labels are assigned by establishing a standard basis for the charged particles at a basepoint x_0. Then a standard path, designated α, is chosen that begins at x_0, winds counterclockwise around the fluxon on the *left*, and returns to x_0. Finally, charged particles are carried around the closed path α, and it is observed that under this parallel transport, the particles are acted upon by $D(a)$, where D is the representation of G according to which the charged particles transform, and $a \in G$ is the particular group element that we assign to the fluxon. Similarly, another standard path, designated β, is chosen that begins at x_0, winds counterclockwise around the fluxon on the *right*, and returns to x_0; the effect of parallel transport around β is found to be $D(b)$, and so the fluxon on the right is labeled with $b \in G$.

Now imagine that a counterclockwise exchange of the two fluxons is performed, after which the calibration procedure is repeated. How will the fluxons be labeled now?

To find the answer, consider the path $\alpha\beta\alpha^{-1}$; here we use α^{-1} to denote the path α traversed in reverse order, and we have adopted the convention that $\alpha\beta\alpha^{-1}$ denotes the path in which α^{-1} is traversed first, followed by β and then α. Now observe that if, as the two fluxons are exchanged counterclockwise, we deform the paths so that they are never crossed by the fluxons, then the path $\alpha\beta\alpha^{-1}$ is deformed to the path α, while the path α is deformed to β:

$$\alpha\beta\alpha^{-1} \mapsto \alpha \ , \quad \alpha \mapsto \beta \ . \tag{9.28}$$



It follows that the effect of transporting a charge around the path α, after the exchange, is equivalent to the effect of transport around the path $\alpha\beta\alpha^{-1}$, before the exchange; similarly, the effect of transport around β, after the exchange, is the same as the effect of transport around α before. We conclude that the braid operator R representing a counterclockwise exchange acts on the fluxes according to

$$R : |a, b\rangle \to |aba^{-1}, a\rangle \ . \tag{9.29}$$

Of course, if the fluxes a and b are commuting elements of G, all the braiding does is swap the positions of the two labels. But if a and b do not commute, the effect of the exchange is more subtle and interesting. The asymmetric form of the action of R is a consequence of our conventions and of the (counterclockwise) sense of the exchange; the inverse operator R^{-1} representing a clockwise exchange acts as

$$R^{-1} : |a, b\rangle \to |b, b^{-1}ab\rangle \ . \tag{9.30}$$

Note that the total flux of the pair of fluxons can be detected by a charged particle that traverses the path $\alpha\beta$ that encloses both members of the pair. Since in principle the charge detecting this total flux could be far, far away, the exchange ought not to alter the total flux; indeed, we find that the product flux ab is preserved by R and by R^{-1}.

The effect of two successive counterclockwise exchanges is the "monodromy" operator R^2, representing the counterclockwise winding of one fluxon about the other, whose action is

$$R^2 : |a, b\rangle \mapsto |(ab)a(ab)^{-1}, (ab)b(ab)^{-1}\rangle \ ; \tag{9.31}$$

both fluxes are conjugated by the total flux ab. That is, winding a counterclockwise about b conjugates b by a (and similarly, winding b clockwise about a conjugates a by b^{-1}). The nontrivial monodromy means that if many fluxons are distributed in the plane, and one of these fluxons is to be brought to my laboratory for analysis, the group element I assign to the fluxon may depend on the path the flux follows as it travels to my lab. If for one choice of path the flux is labeled by $a \in G$, then for other paths any other element bab^{-1} might in principle be assigned. Thus, the conjugacy class in G represented by the fluxon is invariant, but the particular representative of that class is ambiguous.

For example, suppose the group is $G = S_3$, the permutation group on three objects. One of the conjugacy classes contains all of the two-cycle permutations (transpositions of two objects), the three elements $\{(12), (23), (31)\}$. When two such two-cycles fluxes are combined, there are three possibilities for the total flux — the trivial flux e, or one of the three-cycle fluxes (123) or (132). If the total flux is trivial, the braiding of the two fluxes is also trivial (a and $b = a^{-1}$ commute). But if the total flux is nontrivial, then the braid operator R has orbits of length three:

$$\begin{aligned} R &: |(12), (23)\rangle \mapsto |(31), (12)\rangle \mapsto |(23), (31)\rangle \mapsto |(12), (23)\rangle \ , \\ R &: |(23), (12)\rangle \mapsto |(31), (23)\rangle \mapsto |(12), (31)\rangle \mapsto |(23), (12)\rangle \ , \end{aligned} \tag{9.32}$$

Thus, if the two fluxons are exchanged three times, they swap positions (the number of exchanges is odd), yet the labeling of the state is unmodified. This observation means that there can be quantum interference between the "direct" and "exchange" scattering of two fluxons that carry distinct labels in the same conjugacy class, reinforcing the notion that fluxes carrying conjugate labels ought to be regarded as indistinguishable particles.

Since the braid operator acting on pairs of two-cycle fluxes satisfies $R^3 = I$, its eigenvalues are third roots of unity. For example, by taking linear combinations of the three states with total flux (123), we obtain the R eigenstates

$$\begin{aligned} R=1: &\quad |(12),(23)\rangle \ + \ |(31),(12)\rangle \ + \ |(23),(31)\rangle \ , \\ R=\omega: &\quad |(12),(23)\rangle + \bar{\omega}|(31),(12)\rangle + \omega|(23),(31)\rangle \ , \\ R=\bar{\omega}: &\quad |(12),(23)\rangle + \omega|(31),(12)\rangle + \bar{\omega}|(23),(31)\rangle \ , \end{aligned} \tag{9.33}$$

where $\omega = e^{2\pi i/3}$.

Although a pair of fluxes $|a, a^{-1}\rangle$ with trivial total flux has trivial braiding properties, it is interesting for another reason — it carries *charge*. The way to detect the charge of an object is to carry a flux b around the object (counterclockwise); this modifies the object by the action of $D^R(b)$ for some representation R of G. If the charge is zero then the representation is trivial — $D(b) = I$ for all $b \in G$. But if we carry flux b counterclockwise around the state $|a, a^{-1}\rangle$, the state transforms as

$$|a, a^{-1}\rangle \mapsto |bab^{-1}, ba^{-1}b^{-1}\rangle \ , \tag{9.34}$$

a nontrivial action (for at least some b) if a belongs to a conjugacy class with more than one element. In fact, for each conjugacy class α, there is a unique state $|0;\alpha\rangle$ with zero charge, the uniform superposition of the class representatives:

$$|0;\alpha\rangle = \frac{1}{\sqrt{|\alpha|}} \sum_{a\in\alpha} |a, a^{-1}\rangle \ , \tag{9.35}$$

where $|\alpha|$ denotes the order of α. A pair of fluxons in the class α that can be created in a local process must not carry any conserved charges and therefore must be in the state $|0;\alpha\rangle$. Other linear combinations orthogonal to $|0;\alpha\rangle$ carry nonzero charge. This charge carried by a pair of fluxons can be detected by other fluxons, yet oddly the charge cannot be localized on the core of either particle in the pair. Rather it is a collective property of the pair. If two fluxons with a nonzero total charge are brought together, complete annihilation of the pair will be forbidden by charge conservation, even though the total flux is zero.

In the case of a pair of fluxons from the two-cycle class of $G = S_3$, for example, there is a two-dimensional subspace with trivial total flux and nontrivial charge, for which we may choose the basis

$$\begin{aligned} |0\rangle &= |(12),(12)\rangle + \bar{\omega}|(23),(23)\rangle + \omega|(31),(31)\rangle \ , \\ |1\rangle &= |(12),(12)\rangle + \omega|(23),(23)\rangle + \bar{\omega}|(31),(31)\rangle \ . \end{aligned} \tag{9.36}$$

If a flux b is carried around the pair, both fluxes are conjugated by b; therefore the action (by conjugation) of S_3 on these states is

$$\begin{aligned} &D(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad D(23) = \begin{pmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{pmatrix} \ , \quad D(31) = \begin{pmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{pmatrix} \ , \\ &D(123) = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} \ , \quad D(132) = \begin{pmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{pmatrix} \ . \end{aligned} \tag{9.37}$$

This action is just the two-dimensional irreducible representation $R = [2]$ of S_3, and so we conclude that the charge of the pair of fluxons is [2].

Furthermore, under braiding this charge carried by a pair of fluxons can be transferred to other particles. For example, consider a pair of particles, each of which carries charge but no flux (I will refer to such particles as *chargeons*), such that the total charge of the pair is trivial. If one of the chargeons transforms as the unitary irreducible representation R of G, there is a unique conjugate representation $\bar{R}$ that can be combined with R to give the trivial representation; if $\{|R,i\rangle\}$ is a basis for R, then a basis $\{|\bar{R},i\rangle\}$ can be chosen for $\bar{R}$, such that the chargeon pair with trivial charge can be expressed as

$$|0;R\rangle = \frac{1}{\sqrt{|R|}} \sum_i |R,i\rangle \otimes |\bar{R},i\rangle \ . \tag{9.38}$$

Imagine that we create a pair of fluxons in the state $|0;\alpha\rangle$ and also create a pair of chargeons in the state $|0;R\rangle$. Then we wind the chargeon with charge R counterclockwise around the fluxon with flux in class α, and bring the two chargeons together again to see if they will annihilate. What happens?

For a fixed value $a \in \alpha$ of the flux, the effect of the winding on the state of the two chargeons is

$$|0;R\rangle \mapsto \frac{1}{\sqrt{|R|}} \sum_{i,j} |R,j\rangle \otimes |\bar{R},i\rangle D^R_{ji}(a) \ ; \tag{9.39}$$

if the charge of the pair were now measured, the probability that zero total charge would be found is the square of the overlap of this state with $|0;R\rangle$, which is

$$\mathrm{Prob}(0) = \left| \frac{\chi^R(a)}{|R|} \right|^2 \ , \tag{9.40}$$

where

$$\chi^R(a) = \sum_i D^R_{ii}(a) = \text{tr } D^R(a) \tag{9.41}$$

is the *character* of the representation R, evaluated at a. In fact, the character (a trace) is unchanged by conjugation — it takes the same value for all $a \in \alpha$. Therefore, eq. (9.40) is also the probability that the pair of chargeons has zero total charge when one chargeon (initially a member of a pair in the state $|0; R\rangle$) winds around one fluxon (initially a member of a pair in the state $|0; \alpha\rangle$). Of course, since the total charge of all four particles is zero and charge is conserved, after the winding the two pairs have opposite charges — if the pair of chargeons has total charge R', then the pair of fluxons must have total charge $\bar{R}'$, combined with R' to give trivial total charge. A pair of particles with zero total charge and flux can annihilate, leaving no stable particle behind, while a pair with nonzero charge will be unable to annihilate completely. We conclude, then, that if the world lines of a fluxon pair and a chargeon pair link once, the probability that both pairs will be able to annihilate is given by eq. (9.40). This probability is less than one, provided that the representation of R is not one dimensional and the class α is not represented trivially. Thus the linking of the world lines induces an exchange of charge between the two pairs.

For example, in the case where α is the two-cycle class of $G = S_3$ and $R = [2]$ (the two-dimensional irreducible representation of S_3), we see from eq. (9.37) that $\chi^{[2]}(\alpha) = 0$. Therefore, charge is transfered with certainty; after the winding, both the fluxon pair and the chargeon pair transform as $R' = [2]$.

9.10 Superselection sectors of a nonabelian superconductor

In our discussion so far of the nonabelian superconductor, we have been considering two kinds of particles: *fluxons*, which carry flux but no charge, and *chargeons*, which carry charge but no flux. These are not the most general possible particles. It will be instructive to consider what happens when we build a composite particle by combining a fluxon with a chargeon. In particular, what is the charge of the composite? This question is surprisingly subtle; to answer cogently, we should think carefully about how the charge can be measured.

In principle, charge can be measured in an Aharonov-Bohm interference experiment. We could hide the object whose charge is to be found behind a screen in between two slits, shoot a beam of carefully calibrated fluxons at the screen, and detect the fluxons on the other side. From the shift and visibility of the interference pattern revealed by the detected positions of the fluxons, we can determine $D^R(b)$ for each $b \in G$, and so deduce R.

However, there is a catch if the object being analyzed carries a nontrivial flux $a \in G$ as well as charge. Since carrying a flux b around the flux a changes a to bab^{-1}, the two possible paths followed by the b flux do *not* interfere, if a and b do not commute. After the b flux is detected, we could check whether the a flux has been modified, and determine whether the b flux passed through the slit on the left or the slit on the right. Since the flux (a or bab^{-1}) is correlated with the "which way" information (left or right slit), the interference is destroyed.

Therefore, the experiment reveals information about the charge only if a and b commute. Hence the charge attached to a flux a is not described as an irreducible representation of G; instead it is an irreducible representation of a subgroup of G, the normalizer $N(a)$ of a in G, which is defined as

$$N(a) = \{b \in G | ab = ba\} \ . \tag{9.42}$$

The normalizers $N(a)$ and $N(bab^{-1})$ are isomorphic, so we may associate the normalizer with a conjugacy class α of G rather than with a particular element, and denote it as $N(\alpha)$. Therefore, each type of particle that can occur in our nonabelian superconductor really has two labels: a conjugacy class α describing the flux, and an irreducible representation $R^{(\alpha)}$ of $N(\alpha)$ describing the charge. We say that α and $R^{(\alpha)}$ label the *superselection sectors* of the theory, as these are the properties of a localized object that must be conserved in all local physical processes. For particles that carry the labels $(\alpha, R^{(\alpha)})$, it is possible to establish a "bureau of standards" where altogether $|\alpha| \cdot |R^{(\alpha)}| \equiv d_{(\alpha,R^{(\alpha)})}$ different particle species can be distinguished at a particular time and place — this number is called the *dimension* of the sector. But if these particles are braided with other particles the species may change, while the labels $(\alpha, R^{(\alpha)})$ remain invariant.

In any theory of anyons, a dimension can be assigned to each particle type, although as we will see, in general the dimension need not be an integer, and may have no direct interpretation in terms the counting of distinct species of the same type. The *total dimension* $\mathcal{D}$ can be defined by summing over all types; in the case of a nonabelian superconductor we have

$$\mathcal{D}^2 = \sum_\alpha \sum_{R^{(\alpha)}} d^2_{(\alpha,R^{(\alpha)})} = \sum_\alpha |\alpha|^2 \sum_{R^{(\alpha)}} |R^{(\alpha)}|^2 \ . \tag{9.43}$$

Since the sum over the dimension squared for all irreducible representations of a finite group is the order of the group, and the order of the normalizer $N(\alpha)$ is $|G|/|\alpha|$, we obtain

$$\mathcal{D}^2 = \sum_\alpha |\alpha| \cdot |G| = |G|^2 \ ; \tag{9.44}$$

the total dimension is $\mathcal{D} = |G|$.

For the case $G = S_3$ there are 8 particle types, listed here:

Type	Flux	Charge	Dim
A	e	[+]	1
B	e	[-]	1
C	e	[2]	2
D	(12)	[+]	3
E	(12)	[-]	3
F	(123)	[1]	2
G	(123)	$[\omega]$	2
H	(123)	$[\bar{\omega}]$	2

If the flux is trivial (e), then the charge can be any one of the three irreducible representations of S_3 — the trivial one-dimensional representation [+], the nontrivial one-dimensional representation [-], or the two-dimensional representation [2]. If the flux is a two-cycle, then the normalizer group is Z_2, and the charge can be either the trivial representation [+] or the nontrivial representation [-]. And if the flux is a three-cycle, then the normalizer group is Z_3, and the charge can be either the trivial representation [1], the nontrivial representation $[\omega]$, or its conjugate representation $[\bar{\omega}]$. You can verify that the total dimension is $\mathcal{D} = |S_3| = 6$, as expected.

Note that since a commutes with all elements of $N(a)$ by definition, the matrix $D^{R^{(a)}}(a)$ that represents a in the irreducible representation $R^{(a)}$ commutes with all matrices in the representation; therefore by Schur's lemma it is a multiple of the identity:

$$D^{R^{(a)}}(a) = \exp\left(i\theta_{R^{(a)}}\right) I \ . \tag{9.45}$$

To appreciate the significance of the phase $\exp\left(i\theta_{R^{(a)}}\right)$, consider a flux-charge composite in which a chargeon in representation $R^{(a)}$ is bound to the flux a, and imagine rotating the composite object counterclockwise by 2π. This rotation carries the charge around the flux, generating the phase

$$e^{-2\pi i \boldsymbol{J}} = e^{i\theta_{R^{(a)}}} \ ; \tag{9.46}$$

therefore each superselection sector has a definite value of the *topological spin*, determined by $\theta_{R^{(a)}}$.

When two different particle types are fused together, the composite object can be of various types, and the *fusion rules* of the theory specify which types are possible. The flux of the composite can belong to any of the conjugacy classes that can be obtained as a product of representatives of the classes that label the two constituents. Finding the charge of the composite is especially tricky, as we must decompose a tensor product of representations of two different normalizer groups as a sum of representations of the normalizer of the product flux. In the case $G = S_3$, the rule governing the fusion of two particles of type D, for example, is

$$D \times D = A + C + F + G + H \tag{9.47}$$

We have already noted that the fusion of two two-cycle fluxes can yield either a trivial total flux or a three-cycle flux, and that the charge of the composite with trivial total flux can be either [+] or [2]. If the total flux is a three-cycle, then the charge eigenstates are just the braid operator eigenstates that we constructed in eq. (9.33).

For a system of two anyons, why should the eigenstates of the total charge also be eigenstates of the braid operator? We can understand this connection more generally by thinking about the angular momentum of the two-anyon composite object. The monodromy operator R^2 captures the effect of winding one particle counterclockwise around another. This winding is almost the same thing as rotating the composite system counterclockwise by 2π, except that the rotation of the composite system also rotates both of the constituents. We can compensate for the rotation of the constituents by following the counterclockwise rotation of the composite by a clockwise rotation of the constituents. Therefore, the monodromy operator can be expressed as

$$(R^c_{ab})^2 = e^{-2\pi i \boldsymbol{J}_c} e^{2\pi i \boldsymbol{J}_a} e^{2\pi i \boldsymbol{J}_b} = e^{i(\theta_c - \theta_a - \theta_b)} \ . \tag{9.48}$$

Here R^c_{ab} denotes the braid operator for a counterclockwise exchange of particles of types a and b that are combined together into a composite of type c, and we are using a more succinct notation than before, in which a, b, c are complete labels for the superselection sectors (specifying, in the nonabelian superconductor model, both the flux and the charge). Since each superselection sector has a definite topological spin, and the monodromy operator is diagonal in the topological spin basis, we see that eigenstates of the braid operator coincide with charge eigenstates. Note that eq. (9.48) generalizes our earlier observations about abelian anyons — that a composite of two identical anyons has topological spin $e^{i4\theta}$, and that the exchange phase of an anyon-antianyon pair (with trivial total spin) is $e^{-i\theta}$.

9.11 Quantum computing with nonabelian fluxons

A model of anyons is characterized by the answers to two basic questions: (1) What happens when two anyons are combined together (what are the *fusion rules*)? (2) What happens when two anyons are exchanged (what are the *braiding rules*)? We have discussed how these questions are answered in the special case of a nonabelian superconductor model associated with a nonabelian finite group G, and now we wish to see how these fusion and braiding rules can be invoked in a simulation of a quantum circuit.

In formulating the simulation, we will assume these physical capabilities:

Pair creation and identification. We can create pairs of particles, and for each pair we can identify the particle type (the conjugacy class α of the flux of each particle in the pair, and the particles's charge — an irreducible representation $R^{(\alpha)}$ of the flux's normalizer group $N(\alpha)$). This assumption is reasonable because there is no symmetry relating particles of different types; they have distinguishable physical properties — for example, different energy gaps and effective masses. In practice, the only particle types that will be needed are fluxons that carry no charge and chargeons that carry no flux.

Pair annihilation. We can bring two particles together, and observe whether the pair annihilates completely. Thus we obtain the answer to the question: Does this pair of particles have trivial flux and charge, or not? This assumption is reasonable, because if the pair carries a nontrivial value of some conserved quantity, a localized excitation must be left behind when the pair fuses, and this leftover particle is detectable in principle.

Braiding. We can guide the particles along specified trajectories, and so perform exchanges of the particles. Quantum gates will be simulated by choosing particles world lines that realize particular braids.

These primitive capabilities allow us to realize some further derived capabilities that will be used repeatedly. First, we can use the chargeons to calibrate the fluxons and assemble a flux bureau of standards. Suppose that we are presented with two pairs of fluxons in the states $|a, a^{-1}\rangle$ and $|b, b^{-1}\rangle$, and we wish to determine whether the fluxes a and b match or not. We create a chargeon-antichargeon pair, where the charge of the chargeon is the irreducible representation R of G. Then we carry the chargeon around a closed path that encloses the first member of the first fluxon pair and the second member of the second fluxon pair, we reunite the chargeon and antichargeon, and observed whether the chargeon pair annihilates or not. Since the total flux enclosed by the chargeon's path is ab^{-1}, the chargeon pair annihilates with probability

$$\mathrm{Prob}(0) = \left|\frac{\chi^R(ab^{-1})}{|R|}\right|^2 , \tag{9.49}$$

which is less than one if the flux ab^{-1} is not the identity (assuming that the representation R is not one-dimensional and represents ab^{-1} nontrivially). Thus, if annihilation of the chargeon pair does not occur, we know for sure that a and b are distinct fluxes, and each time annihilation does occur, it becomes increasingly likely that a and b are equal. By repeating this procedure a modest number of times, we can draw a conclusion about whether a and b are the same, with high statistical confidence.

This procedure allows us to sort the fluxon pairs into bins, where each pair in a bin has the same flux. If a bin contains n pairs, its state is, in general, a mixture of states of the form

$$\sum_{a\in G} \psi_a |a, a^{-1}\rangle^{\otimes n} \ . \tag{9.50}$$

By discarding just one pair in the bin, each such state becomes a mixture

$$\sum_{a\in g} \rho_a \left(|a\rangle\langle a|\right)^{\otimes(n-1)} \ ; \tag{9.51}$$

we may regard each bin as containing $(n-1)$ pairs, all with the same definite flux, but where that flux is as yet unknown.

Which bin is which? We want to label the bins with elements of G. To arrive at a consistent labeling, we withdraw fluxon pairs from three different bins. Suppose the three pairs are $|a, a^{-1}\rangle$, $|b, b^{-1}\rangle$, and $|c, c^{-1}\rangle$, and that we want to check whether $c = ab$. We create a chargeon-antichargeon pair, carry the chargeon around a closed path that encloses the first member of the first fluxon pair, the first member of the second fluxon pair, and second member of the third fluxon pair, and observe whether the reunited chargeon pair annihilates or not. Since the total flux enclosed by the chargeon's path is abc^{-1}, by repeating this procedure we can determine with high statistical confidence whether ab and c are the same. Such observations allow us to label the bins in some manner that is consistent with the group composition rule. This labeling is unique apart from group automorphisms (and ambiguities arising from any automorphisms may be resolved arbitrarily).

Once the flux bureau of standards is established, we can use it to measure the unknown flux of an unlabeled pair. If the state of the pair to be measured is $|d, d^{-1}\rangle$, we can withdraw the labeled pair $|a, a^{-1}\rangle$ from a bin, and use chargeon pairs to measure the flux ad^{-1}. By repeating this procedure with other labeled fluxes, we can eventually determine the value of the flux d, realizing a projective measurement of the flux.

For a simulation of a quantum circuit using fluxons, we will need to perform logic gates that act upon the value of the flux. The basic gate we will use is realized by winding counterclockwise a fluxon pair with state

$|a, a^{-1}\rangle$ around the first member of another fluxon pair with state $|b, b^{-1}\rangle$. Since the $|a, a^{-1}\rangle$ pair has trivial total flux, the $|b, b^{-1}\rangle$ pair is unaffected by this procedure. But since in effect the flux b travels counterclockwise about both members of the pair whose initial state was $|a, a^{-1}\rangle$, this pair is transformed as

$$|a, a^{-1}\rangle \mapsto |bab^{-1}, ba^{-1}b^{-1}\rangle \ . \tag{9.52}$$

We will refer to this operation as the *conjugation gate* acting on the fluxon pair.

To summarize what has been said so far, our primitive and derived capabilities allow us to: (1) Perform a projective flux measurement, (2) perform a destructive measurement that determines whether or not the flux and charge of a pair is trivial, and (3) execute a conjugation gate. Now we must discuss how to simulate a quantum circuit using these capabilities.

The next step is to decide how to encode qubits using fluxons. Appropriate encodings can be chosen in many ways; we will stick to one particular choice that illustrates the key ideas — namely we will encode a qubit by using a pair of fluxons, where the total flux of the pair is trivial. We select two noncommuting elements $a, b \in G$, where $b^2 = e$, and choose a computational basis for the qubit

$$|\bar{0}\rangle = |a, a^{-1}\rangle \ , \quad |\bar{1}\rangle = |bab^{-1}, ba^{-1}b^{-1}\rangle \ . \tag{9.53}$$

The crucial point is that a single isolated fluxon with flux a looks identical to a fluxon with the conjugate flux bab^{-1}. Therefore, if the two fluxons in a pair are kept far apart from one another, local interactions with the environment will not cause a superposition of the states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ to decohere. The quantum information is protected from damage because it is stored nonlocally, by exploiting a topological degeneracy of the states where the fluxon and antifluxon are pinned to fixed and distantly separated positions.

However, in contrast with the topological degeneracy that arises in systems with abelian anyons, this protected qubit can be measured relatively easily, without resorting to delicate interferometric procedures that extract Aharonov-Bohm phases. We have already described how to measure flux using previously calibrated fluxons; therefore we can perform a projective measurement of the encoded Pauli operator $\bar{Z}$ (a projection onto the basis $\{|\bar{0}\rangle, |\bar{1}\rangle\}$). We can also measure the complementary Pauli operator $\bar{X}$, albeit destructively and imperfectly. The $\bar{X}$ eigenstates are

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}\rangle \pm |\bar{1}\rangle\right) \equiv \frac{1}{\sqrt{2}}\left(|a, a^{-1}\rangle \pm |bab^{-1}, ba^{-1}b^{-1}\rangle\right) \ ; \tag{9.54}$$

therefore the state $|-\rangle$ is *orthogonal* to the zero-charge state

$$|0;\alpha\rangle = \frac{1}{\sqrt{|\alpha|}}\left(\sum_{c\in\alpha}|c,c^{-1}\rangle\right) , \tag{9.55}$$

where α is the conjugacy class that contains a. On the other hand, the state $|+\rangle$ has a nonzero overlap with $|0;\alpha\rangle$

$$\langle +|0;\alpha\rangle = \sqrt{2/|\alpha|} ; \tag{9.56}$$

Therefore, if the two members of the fluxon pair are brought together, complete annihilation is impossible if the state of the pair is $|-\rangle$, and occurs with probability Prob(0) $= 2/|\alpha|$ if the state is $|+\rangle$.

Note that it is also possible to prepare a fluxon pair in the state $|+\rangle$. One way to do that is to create a pair in the state $|0;\alpha\rangle$. If α contains only the two elements a and bab^{-1} we are done. Otherwise, we compare the newly created pair with calibrated pairs in each of the states $|c,c^{-1}\rangle$, where $c\in\alpha$ and c is distinct from both a and bab^{-1}. If the pair fails to match any of these $|c,c^{-1}\rangle$ pairs, its state must be $|+\rangle$.

To go further, we need to characterize the computational power of the conjugation gate. Let us use a more compact notation, in which the state $|x,x^{-1}\rangle$ of a fluxon pair is simply denoted $|x\rangle$, and consider the transformations of the state $|x,y,z\rangle$ that can be built from conjugation gates. By winding the third pair through the first, either counterclockwise or clockwise, we can execute the gates

$$|x,y,z\rangle \mapsto |x,y,xzx^{-1}\rangle , \quad |x,y,z\rangle \mapsto |x,y,x^{-1}zx\rangle , \tag{9.57}$$

and by winding the third pair through the second, either counterclockwise or clockwise, we can execute

$$|x,y,z\rangle \mapsto |x,y,yzy^{-1}\rangle , \quad |x,y,z\rangle \mapsto |x,y,y^{-1}zy\rangle ; \tag{9.58}$$

furthermore, by borrowing a pair with flux $|c\rangle$ from the bureau of standards, we can execute

$$|x,y,z\rangle \mapsto |x,y,czc^{-1}\rangle \tag{9.59}$$

for any constant $c\in G$. Composing these elementary operations, we can execute any gate of the form

$$|x,y,z\rangle \mapsto |x,y,fzf^{-1}\rangle , \tag{9.60}$$

where the function $f(x,y)$ can be expressed in *product form* — that is, as a finite product of group elements, where the elements appearing in the product may be the inputs x and y, their inverses x^{-1} and y^{-1}, or constant elements of G, each of which may appear in the product any number of times.

What are the functions $f(x, y)$ that can be expressed in this form? The answer depends on the structure of the group G, but the following characterization will suffice for our purposes. Recall that a subgroup H of a finite group G is *normal* if for any $h \in H$ and any $g \in G$, $ghg^{-1} \in H$, and recall that a finite group G is said to be *simple* if G has no normal subgroups other than G itself and the trivial group $\{e\}$. It turns out that if G is a simple nonabelian finite group, then *any* function $f(x, y)$ can be expressed in product form. In the computer science literature, a closely related result is often called *Barrington's theorem*.

In particular, then, if the group G is a nonabelian simple group, there is a function f realizable in product form such that

$$f(a, a) = f(a, bab^{-1}) = f(bab^{-1}, a) = e \ , \quad f(bab^{-1}, bab^{-1}) = b \ . \tag{9.61}$$

Thus for $x, y, z \in \{a, bab^{-1}\}$, the action eq. (9.60) causes the flux of the third pair to "flip" if and only if $x = y = bab^{-1}$; we have constructed from our elementary operations a Toffoli gate in the computational basis. Therefore, conjugation gates suffice for universal reversible *classical* computation acting on the standard basis states.

The nonabelian simple group of minimal order is A_5, the group of even permutations of five objects, with $|A_5| = 60$. Therefore, one concrete realization of universal classical computation using conjugation gates is obtained by choosing a to be the three-cycle element $a = (345) \in A_5$, and b to be the product of two-cycles $b = (12)(34) \in A_5$, so that $bab^{-1} = (435)$.

With this judicious choice of the group G, we achieve a topological realization of universal classical computation, but how can be go still further, to realize universal quantum computation? We have the ability to prepare computational basis states, to measure in the computational basis, and to execute Toffoli gates, but these tools are entirely classical. The only nonclassical tricks at our disposal are the ability to prepare $\bar{X} = 1$ eigenstates, and the ability to perform an imperfect destructive measurement of $\bar{X}$. Fortunately, these additional capabilities are sufficient.

In our previous discussions of quantum fault tolerance, we have noted that if we can do the classical gates Toffoli and CNOT, it suffices for universal quantum computation to be able to apply each of the Pauli operators X, Y, and Z, and to be able to perform projective measurements of each of X, Y, and Z. We already know how to apply the classical gate X and to measure Z (that is, project onto the computational basis). Projective measurement of X and Y, and execution of Z, are still missing from our repertoire. (Of course, if we can apply X and Z, we can also apply their product $ZX = iY$.)

Next, let's see how to elevate our imperfect destructive measurement of X to a reliable projective measurement of X. Recall the action by conjugation of a CNOT on Pauli operators:

$$\mathrm{CNOT} : XI \mapsto XX \ , \tag{9.62}$$

where the first qubit is the control and the second qubit is the target of the CNOT. Therefore, CNOT gates, together with the ability to prepare $X = 1$ eigenstates and to perform destructive measurements of X, suffice to realize projective measurements of X. We can prepare an ancilla qubit in the $X = 1$ eigenstate, perform a CNOT with the ancilla as control and the data to be measured as target, and then measure the ancilla destructively. The measurement prepares the data in an eigenstate of X, whose eigenvalue matches the outcome of the measurement of the ancilla. In our case, the destructive measurement is not fully reliable, but we can repeat the measurement multiple times. Each time we prepare and measure a fresh ancilla, and after a few repetitions, we have acceptable statistical confidence in the inferred outcome of the measurement.

Now that we can measure X projectively, we can prepare $X = -1$ eigenstates as well as $X = 1$ eigenstates (for example, we follow a Z measurement with an X measurement until we eventually obtain the outcome $X = -1$). Then, by performing a CNOT gate whose target is an $X = -1$ eigenstate, we can realize the Pauli operator Z acting on the control qubit. It only remains to show that a measurement of Y can be realized.

Measurement of Y seems problematic at first, since our physical capabilities have not provided any means to distinguish between $Y = 1$ and $Y = -1$ eigenstates (that is, between a state ψ and its complex conjugate ψ^*). However, this ambiguity actually poses no serious difficulty, because it makes no difference how the ambiguity is resolved. Were we to replace measurement of Y by measurement of $-Y$ in our simulation of a unitary transformation U, the effect would be that U^* is simulated instead; this replacement would not alter the probability distributions of outcomes for measurements in the standard computational basis.

To be explicit, we can formulate a protocol for measuring Y by noting first that applying a Toffoli gate whose target qubit is an $X = -1$ eigenstate realizes the controlled-phase gate $\Lambda(Z)$ acting on the two control qubits. By composing this gate with the CNOT gate $\Lambda(X)$, we obtain the gate $\Lambda(iY)$ acting as

$$\begin{aligned}\Lambda(iY): \quad & |X=+1\rangle \otimes |Y=+1\rangle \mapsto |Y=+1\rangle \otimes |Y=+1\rangle \ ,\\ & |X=+1\rangle \otimes |Y=-1\rangle \mapsto |Y=-1\rangle \otimes |Y=-1\rangle \ ,\\ & |X=-1\rangle \otimes |Y=+1\rangle \mapsto |Y=-1\rangle \otimes |Y=+1\rangle \ ,\\ & |X=-1\rangle \otimes |Y=-1\rangle \mapsto |Y=+1\rangle \otimes |Y=-1\rangle \ , \end{aligned} \tag{9.63}$$

where the first qubit is the control and the second is the target. Now suppose that my trusted friend gives me just one qubit that he assures me has been prepared in the state $|Y = 1\rangle$. I know how to prepare $|X = 1\rangle$ states myself and I can execute $\Lambda(iY)$ gates; therefore since a $\Lambda(iY)$ gate with $|Y = 1\rangle$ as its target transforms $|X = 1\rangle$ to $|Y = 1\rangle$, I can make many copies of the $|Y = 1\rangle$ state I obtained from my friend. When I wish to measure Y, I apply the inverse of $\Lambda(iY)$, whose target is the qubit to be measured, and whose control is one of my $Y = 1$ states; then I perform an X measurement of the ancilla to read out the result of the Y measurement of the other qubit.

What if my friend lies to me, and gives me a copy of the state $|Y = -1\rangle$ instead? Then I'll make many copies of the $|Y = -1\rangle$ state, and I will be measuring $-Y$ when I think I am measuring Y. My simulation will work just the same as before; I'll actually be simulating the complex conjugate of the ideal circuit, but that won't change the final outcome of the quantum computation. If my friend flipped a coin to decide whether to give me the $|Y = 1\rangle$ state or the $|Y = -1\rangle$, this too would have no effect on the fidelity of my simulation. Therefore, it turns out I don't need by friend's help at all — instead of using the $|Y = 1\rangle$ state I would have received from him, I may use the random state $\rho = I/2$ (an equally weighted mixture of $|Y = 1\rangle$ and $|Y = -1\rangle$, which I know how to prepare myself).

This completes the demonstration that we can simulate a quantum circuit efficiently and fault tolerantly using the fluxons and chargeons of a nonabelian superconductor, at least in the case where G is a simple nonabelian finite group.[§] Viewed as a whole, including all state preparation and calibration of fluxes, the simulation can be described this way: Many pairs of anyons (fluxons and chargeons) are prepared, the anyon world lines follow a particular braid, and pairs of anyons are fused to see whether they will annihilate. The simulation is nondeterministic in the sense that the actual braid executed by the anyons depends on the outcomes of measurements performed (via fusion) during the course of the simulation. It is robust if the temperature is low compared to the energy gap, and if particles are kept sufficiently far apart from one another (except when pairs are being created and fused), to suppress the exchange of virtual anyons. Small deformations in the world lines of the particles have no effect on the outcome of the computation, as long as the braiding of the particles is in the correct topological class.

[§] Mochon has shown that universal quantum computation is possible for a larger class of groups.

9.12 Anyon models generalized

Our discussion of the nonabelian superconductor model provides an existence proof for fault-tolerant quantum computation using anyons. But the model certainly has drawbacks. The scheme we described lacks beauty, elegance, or simplicity.

I have discussed this model in such detail because it is rather concrete and so helps us to build intuition about the properties of nonabelian anyons. But now that we understand better the key concepts of braiding and fusing in anyon models, we are ready to start thinking about anyons in a more general and abstract way. Our new perspective will lead us to new models, including some that are far simpler than those we have considered so far. We will be able to jettison much of the excess baggage that burdened the nonabelian superconductor model, such as the distinction between fluxons and chargeons, the calibration of fluxes, and the measurements required to simulate nonclassical gates. The simpler models we will now encounter are more naturally conducive to fault-tolerant computing, and more plausibly realizable in reasonable physical systems.

A model of anyons is a theory of particles on a two-dimensional surface (which we will assume to be the plane), where the particles carry locally conserved *charges*. We also assume that the theory has a mass gap, so that there are no long-range interactions between particles mediated by massless particles. The model has three defining properties:

1. A list of particle *types*. The types are labels that specify the possible values of the conserved charge that a particle can carry.

2. Rules for *fusing* and *splitting*, which specify the possible values of the charge that can be obtained when two particles of known charge are combined together, and the possible ways in which the charge carried by a single particle can be split into two parts.

3. Rules for *braiding*, which specify what happens when two particles are exchanged (or when one particle is rotated by 2π).

Let's now discuss each of these properties in more detail.

9.12.1 Labels

I will use Latin letters $\{a, b, c, \ldots\}$ for the labels that distinguish different types of particles. (For the case of the nonabelian superconductor, the label was $(\alpha, R^{(\alpha)})$, specifying a conjugacy class and an irreducible representation of the normalizer of the class, but now our notation will be more compact). We will assume that the set of possible labels is finite. The symbol a represents the value of the conserved charge carried by the particle. Sometimes we say that this label specifies a *superselection sector* of the theory. This term just means that the label a is a property of a localized object that cannot be changed by any local physical process. That is, if one particle is at all times well isolated from other particles, its label will never change. In particular, local interactions between the particle and its environment may jostle the particle, but will not alter the label. This local conservation of charge is the essential reason that anyons are amenable to fault-tolerant quantum information processing.

There is one special label — the identity label 1. A particle with the label 1 is really the same thing as no particle at all. Furthermore, for each particle label a there is a conjugate label $\bar{a}$, and there is a *charge conjugation* operation C (where $C^2 = I$) acting on the labels that maps a label to its conjugate:

$$C : a \mapsto \bar{a} \mapsto a \ . \tag{9.64}$$

It is possible for a label to be self-conjugate, so that $\bar{a} = a$. For example, $\bar{1} = 1$.

We will want to consider states of n particles, where the particles have a specified order. Therefore, it is convenient to imagine that the particles are arranged on a particular line (such as the real axis) from left to right in consecutive order. The n particles are labeled $(a_1, a_2, a_3 \ldots, a_n)$, where a_1 is attached to the particle furthest to the left, a_n to the particle furthest to the right.

9.12.2 Fusion spaces

When two particles are combined together, the composite object also has a charge. The *fusion rules* of the model specify the possible values of the total charge c when the constituents have charges a and b. These can be written

$$a \times b = \sum_c N_{ab}^c \ c \ , \tag{9.65}$$

where each N_{ab}^c is a nonnegative integer and the sum is over the complete set of labels. Note that a, b and c are labels, not vector spaces; the product on the left-hand side is not a tensor product and the sum on the right-hand side is not a direct sum. Rather, the fusion rules can be regarded as an abstract relation on the label set that maps the ordered triple $(a, b; c)$ to N_{ab}^c. This relation is symmetric in a and b ($a \times b = b \times a$) — the possible charges of the composite do not depend on whether a is on the left or the right. Read backwards, the fusion rules specify the possible ways for the charge c to split into two parts with charges a and b.

If $N_{ab}^c = 0$, then charge c cannot be obtained when we combine a and b. If $N_{ab}^c = 1$, then c *can* be obtained — in a unique way. If $N_{ab}^c > 1$, then c can be obtained in N_{ab}^c *distinguishable ways*. The notion that fusing two charges can yield a third charge in more than one possible way should be familiar from group representation theory. For example, the rule governing the fusion of two octet representations of SU(3) is

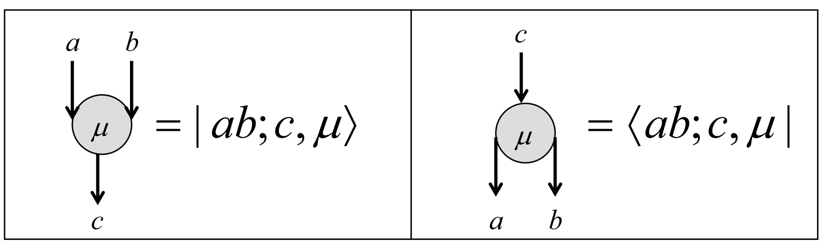
$$8 \times 8 = 1 + 8 + 8 + 10 + \overline{10} + 27 \ , \tag{9.66}$$

so that $N_{88}^8 = 2$. We emphasize again, however, that while the fusion rules for group representations can be interpreted as a decomposition of a tensor product of vector spaces as a direct sum of vector spaces, in general the fusion rules in an anyon model have no such interpretation.

The N_{ab}^c distinguishable ways that c can arise by fusing a and b can be regarded as the orthonormal basis states of a Hilbert space V_{ab}^c. We call V_{ab}^c a *fusion space* and the states it contains *fusion states*. The basis elements for V_{ab}^c may be denoted

$$\{|ab; c, \mu\rangle \ , \quad \mu = 1, 2, \ldots, N_{ab}^c\} \ . \tag{9.67}$$

It is quite convenient to introduce a graphical notation for the fusion basis states:



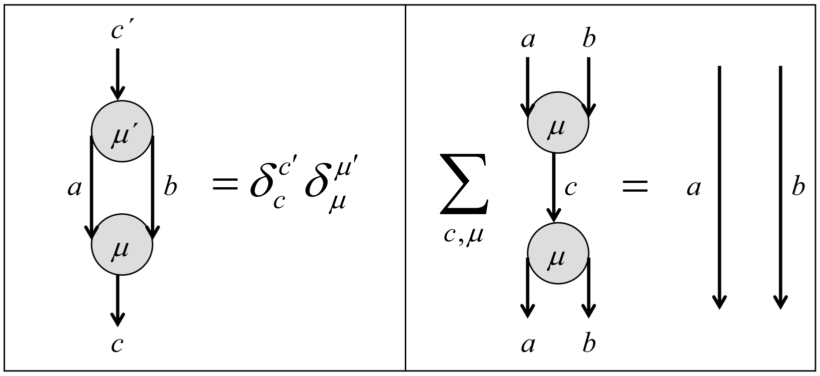
The state $|ab; c, \mu\rangle$ is represented as a circle containing the symbol μ; connected to the circle are lines labeled a and b with incoming arrows, representing the charges being fused, and a line labeled c with an outgoing arrow, representing the result of the fusion. There is a dual vector space V_c^{ab} describing the states that arise when charge c splits into charges a and b, and a dual basis with the sense of the arrow reversed (c coming in, a and b going out). The spaces V_{ab}^c with different values of c are mutually orthogonal, so that the fusion basis elements satisfy

$$\langle ab; c'\mu' | ab; c, \mu\rangle = \delta_c^{c'} \delta_\mu^{\mu'} \ , \tag{9.68}$$

and the completeness of the fusion basis can be expressed as

$$\sum_{c,\mu} |ab; c, \mu\rangle\langle ab; c, \mu| = I_{ab} \ , \tag{9.69}$$

where I_{ab} denotes the projector onto the space $\oplus_c V_{ab}^c$, the full Hilbert space for the anyon pair ab.



There are some natural isomorphisms among fusion spaces. First of all, $V_{ab}^c \cong V_{ba}^c$; these vector spaces are associated with different labelings of the two particles (if $a \neq b$) and so should be regarded as distinct, but they are isomorphic spaces because fusion is symmetric. We may also "raise and lower indices" of a fusion space by replacing a label by its conjugate, *e.g.*,

$$V_{ab}^c \cong V_{a\bar{c}}^{\bar{b}} \cong V_{ab\bar{c}}^1 \cong V_a^{\bar{b}c}, \cong V_{\bar{c}}^{\bar{a}\bar{b}} \cong \cdots ; \tag{9.70}$$

in the diagrammatic notation, we have the freedom to reverse the sense of a line while conjugating the line's label. The space $V_{ab\bar{c}}^1$, represented as a diagram with three incoming lines, is the space spanned by the distinguishable ways to obtain the trivial total charge 1 when fusing three particles with labels a, b, $\bar{c}$.

The charge 1 deserves its name because it fuses trivially with other particles:

$$a \times 1 = a\ . \tag{9.71}$$

Because of the isomorphism $V_{a1}^a \cong V_{a\bar{a}}^1$, we conclude that $\bar{a}$ is the unique label that can fuse with a to yield 1, and that this fusion can occur in only one way. Similarly, $V_{a1}^a \cong V_1^{a\bar{a}}$ means that pairs of particles created out of the vacuum have conjugate charges.

An anyon model is *nonabelian* if

$$\dim\left(\bigoplus_c V_{ab}^c\right) = \sum_c N_{ab}^c \geq 2 \tag{9.72}$$

for at least some pair of labels ab; otherwise the model is *abelian*. In an abelian model, any two particles fuse in a unique way, but in a nonabelian model, there are some pairs of particles that can fuse in more than one way, and there is a Hilbert space of two or more dimensions spanned by these distinguishable states. We will refer to this space as the "topological

Hilbert space" of the pair of anyons, to emphasize that this quantum information is encoded nonlocally — it is a collective property of the pair, not localized on either particle. Indeed, when the two particles with labels a and b are far apart, different states in the topological Hilbert space look identical locally. Therefore, this quantum information is well hidden, and invulnerable to decoherence due to local interactions with the environment.

It is for this reason that we propose to use nonabelian anyons in the operation of a quantum computer. Of course, nonlocally encoded information is not only hidden from the environment; we are unable to read it ourselves as well. However, with nonabelian anyons, we can have our cake and eat it too! At the conclusion of a quantum computation, when we are ready to perform the readout, we can bring the anyons together in pairs and observe the result of this fusion. In fact, it will suffice to distinguish the case where the charge of the composite is $c = 1$ from the case $c \neq 1$ — that is, to distinguish a residual particle (unable to decay because of its nontrivial conserved charge) from no particle at all.

Note that for each pair of anyons this topological Hilbert space is finite-dimensional. An anyon model with this property is said to be *rational*. As in our discussion of the topologically degenerate ground state for an abelian model, anyons in rational nonabelian models always have topological spins that are roots of unity.

9.12.3 Braiding: the R-matrix

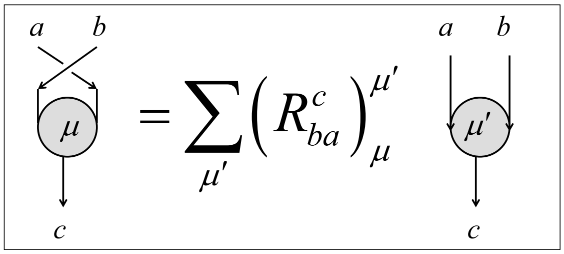
When two particles with labels a and b undergo a counterclockwise exchange, their total charge c is unchanged. Therefore, since the two particles swap positions on the line, the swap induces a natural isomorphism mapping the Hilbert space V_{ba}^c to V_{ab}^c; this map is the braid operator

$$R : V_{ba}^c \to V_{ab}^c \ . \tag{9.73}$$

If we choose canonical bases $\{|ba; c, \mu\rangle\}$ and $\{|ab; c, \mu'\rangle\}$ for these two spaces, R can be expressed as the unitary matrix

$$R : |ba; c, \mu\rangle \mapsto \sum_{\mu'} |ab; c, \mu'\rangle \, (R_{ab}^c)_{\mu}^{\mu'} \ \ ; \tag{9.74}$$

note that R may have a nontrivial action on the fusion states. When we represent the action of R diagrammatically, it is convenient to fix the positions of the labels a and b on the incoming lines, and twist the lines counterclockwise as they move toward the fusion vertex (μ)— the graph with twisted lines represents the state in V_{ab}^c obtained by applying R to $|ba; c, \mu\rangle$, which can be expanded in terms of the canonical basis for V_{ab}^c:



The *monodromy* operator

$$R^2 : V_{ab}^c \to V_{ab}^c \tag{9.75}$$

is an isomorphism from V_{ab}^c to itself, representing the effect of winding a counterclockwise around b. As we already remarked in our discussion of the nonabelian superconductor, the monodromy operator is equivalent to rotating c by 2π while rotating a and b by -2π; therefore, the eigenvalues of the monodromy operator are determined by the *topological spins* of the particles:

$$\left(R_{ab}^c\right)^2 = e^{-2\pi i J_c} e^{2\pi i J_a} e^{2\pi i J_b} \equiv e^{i(\theta_c - \theta_a - \theta_b)} \ . \tag{9.76}$$

Furthermore, as we argued for the case of abelian anyons, the topological spin is determined by the braid operator acting on a particle-antiparticle pair with trivial total charge:

$$e^{-i\theta_a} = R_{a\bar{a}}^1 \tag{9.77}$$

(because creating a pair, exchanging, and annihilating is equivalent to rotating the particle by -2π).

9.12.4 Associativity of fusion: the F-matrix

Fusion is associative:

$$(a \times b) \times c = a \times (b \times c) \ . \tag{9.78}$$

Mathematically, this is an axiom satisfied by the fusion rules of an anyon model. Physically, it is imposed because the total charge of a system of three particles is an intrinsic property of the three particles, and ought not to depend on whether we first fuse a and b and then fuse the result with c, or first fuse b and c and then fuse the result with a.

Therefore, when three particles with charges a, b, c are fused to yield a total charge of d, there are two natural ways to decompose the topological Hilbert space in terms of the fusion spaces of pairs of particles:

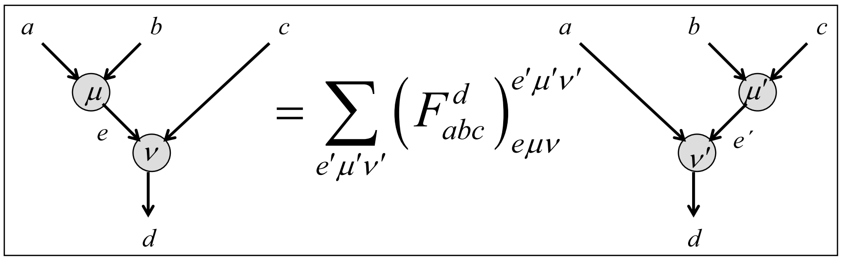
$$V_{abc}^d \cong \bigoplus_e V_{ab}^e \otimes V_{eb}^d \cong \bigoplus_{e'} V_{ae'}^d \otimes V_{bc}^{e'} \ . \tag{9.79}$$

Correspondingly, there are two natural orthonormal bases for V^d_{abc}, which we may denote

$$
\begin{aligned}
&|(ab)c \to d; e\mu\nu\rangle \equiv |ab; e, \mu\rangle \otimes |ec; d, \nu\rangle \ , \\
&|a(bc) \to d; e'\mu'\nu'\rangle \equiv |ae'; d, \nu'\rangle \otimes |bc; e', \mu'\rangle \ ,
\end{aligned}
\tag{9.80}
$$

and which are related by a unitary transformation F:

$$
|(ab)c \to d; e\mu\nu\rangle = \sum_{e'\mu'\nu'} |a(bc) \to d; e'\mu'\nu'\rangle \left(F^d_{abc}\right)^{e'\mu'\nu'}_{e\mu\nu} \ . \tag{9.81}
$$



The unitary matrices F^d_{abc} are sometimes called *fusion matrices*; however, rather than risk causing confusion between F and the fusion *rules* N^c_{ab}, I will just call it the F-*matrix*.

9.12.5 Many anyons: the standard basis

In an anyonic quantum computer, we process the topological quantum state of n anyons by braiding the anyons. For describing this computation, it is convenient to adopt a standard basis for such a Hilbert space.

Suppose that n anyons with total charge c, arranged sequentially along a line, carry labels $a_1, a_2, a_3, \ldots, a_n$. Imagine fusing anyons 1 and 2, then fusing the result with anyon 3, then fusing the result with anyon 4, and so on. Associated with fusion in this order is a decomposition of the topological Hilbert space of the n anyons

$$
V^c_{a_1a_2a_3\cdots a_n} \cong \bigoplus_{b_1,b_2,\ldots,b_{n-2}} V^{b_1}_{a_1a_2} \otimes V^{b_2}_{b_1a_3} \otimes V^{b_3}_{b_2a_4} \otimes \cdots \otimes V^c_{b_{n-2}a_n} \ . \tag{9.82}
$$

Note that this space does *not* have a natural decomposition as a tensor product of subsystems associated with the localized particles; rather, we have expressed it as a direct sum of many tensor products. For nonabelian anyons, its dimension

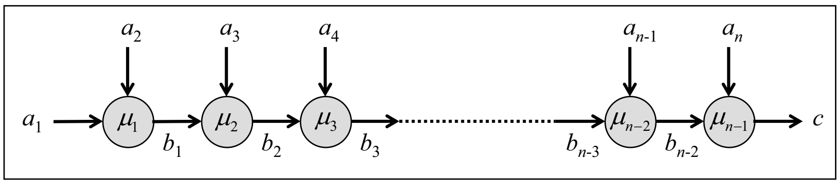
$$
\begin{aligned}
&\dim\left(V^c_{a_1a_2a_3\cdots a_n}\right) \equiv N^c_{a_1a_2a_3\cdots a_n} \\
= &\sum_{b_1,b_2,b_3,\ldots b_{n-2}} N^{b_1}_{a_1a_2} N^{b_2}_{b_1a_3} N^{b_3}_{b_2a_4} \ldots N^c_{b_{n-2}a_n}
\end{aligned}
\tag{9.83}
$$

is exponential in n; thus the topological Hilbert space is a suitable arena for powerful quantum information processing.

This decomposition of $V^c_{a_1 a_2 a_3 \cdots a_n}$ suggests a standard basis whose elements are labeled by the intermediate charges $b_1, b_2, \ldots b_{n-2}$ and by the basis elements $\{|\mu_j\rangle\}$ for the fusion spaces $V^{b_j}_{b_{j-1}, a_{j+1}}$:

$$\{|a_1 a_2; b_1, \mu_1\rangle |b_1 a_3; b_2, \mu_2\rangle \cdots |b_{n-3} a_{n-1}; b_{n-2}, \mu_{n-2}\rangle |b_{n-2} a_n; c, \mu_{n-1}\rangle\}\ , \tag{9.84}$$

or in diagrammatic notation:



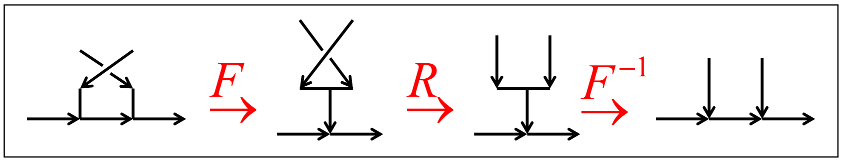
Of course, this basis is chosen arbitrarily. If we preferred, we could imagine fusing the particles in a different order, and would obtain a different basis that can be expressed in terms of our standard one with help from the F-matrix.

9.12.6 Braiding in the standard basis: the B-matrix

We would like to consider what happens to states of the topological vector space $V^c_{a_1 a_2 a_3 \cdots a_n}$ of n anyons when the particles are exchanged with one another. Actually, since exchanges can swap the positions of particles with distinct labels, they may map one topological vector space to another by permuting the labels. Nevertheless, we can consider the direct sums of the vector spaces associated with all the possible permutations of the labels, which will provide a representation of the braid group B_n.

We would like to describe how this representation acts on the standard bases for these spaces. It suffices to say how exchanges of neighboring particles are represented; that is, to specify the action of the generators of the braid group. However, so far, we have discussed only the action of the braid group on a pair of particles with definite total charge (the R-matrix), which is not in itself enough to tell us its action on the standard bases.

The way out of this quandary is to observe that, by applying the F-matrix, we can move from the standard basis to the basis in which the R-matrix is block diagonal, apply R, and then apply F^{-1} to return to the standard basis:



The composition of these three operations, which expresses the effect of braiding in the standard basis, is denoted B and sometimes called the "braid matrix;" but to avoid confusion between B and R, I will just call it the B-*matrix*.

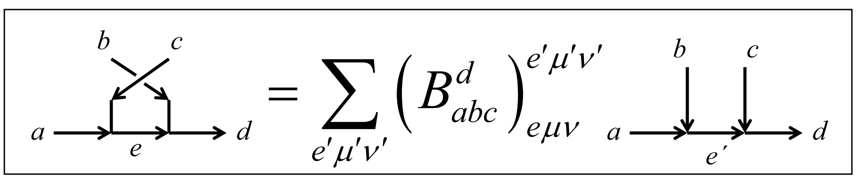
Consider exchanging the anyons in positions j and $j+1$ along the line. In our decomposition of $V^c_{a_1 a_2 a_3 \cdots a_n}$, this exchange acts on the space

$$V^{b_j}_{b_{j-2},a_j,a_{j+1}} = \bigoplus_{b_{j-1}} V^{b_{j-1}}_{b_{j-2},a_j} \otimes V^{b_j}_{b_{j-1},a_{j+1}} \ . \tag{9.85}$$

To reduce the number of subscripts, we will call this space V^d_{acb}, which is transformed by the exchange as

$$B : V^d_{acb} \to V^d_{abc} \ . \tag{9.86}$$

Let us express the action of B in terms of the standard bases for the two spaces V^d_{acb} and V^d_{abc}.



To avoid cluttering the equations, I suppress the labels for the fusion space basis elements (it is obvious where they should go). Hence we write

$$\begin{aligned} B|(ac)b \to d; e\rangle &= \sum_f B|a(cb) \to d; f\rangle \left(F^d_{acb}\right)^f_e \\ &= \sum_f |a(bc) \to d; f\rangle R^f_{bc} \left(F^d_{acb}\right)^f_e \\ &= \sum_{f,g} |(ab)c \to d; g\rangle \left[\left(F^{-1}\right)^d_{abc}\right]^g_f R^f_{bc} \left(F^d_{acb}\right)^f_e \ , \end{aligned} \tag{9.87}$$

or

$$B : |(ac)b \to d; e\rangle \mapsto \sum_g |(ab)c \to d; g\rangle \left(B^d_{abc}\right)^g_e \ , \tag{9.88}$$

where

$$\left(B^d_{abc}\right)^g_e = \sum_f \left[\left(F^{-1}\right)^d_{abc}\right]^g_f R^f_{bc} \left(F^d_{acb}\right)^f_e \ . \tag{9.89}$$

We have expressed the action of the B-matrix in the standard basis in terms of the F-matrix and R-matrix, as desired.

Thus, the representation of the braid group realized by n anyons is completely characterized by the F-matrix and the R-matrix. Furthermore, we have seen that the R matrix also determines the topological spins of the anyons, so that we have actually constructed a representation of a larger group whose generators include both the exchanges of neighboring particles *and* 2π rotations of the particles. A good name for this group would be the *ribbon* group, as its elements are in one-to-one correspondence with the topological classes of braided ribbons (which can be twisted) rather than braided strings; however, mathematicians have already named it "the mapping class group for the sphere with n punctures."

And with that observation we have completed our description of an anyon model in this general setting. The model is specified by: (1) a label set, (2) the fusion rules, (3) the R-matrix, and (4) the F matrix.

The mathematical object we have constructed is called a *unitary topological modular functor*, and it is closely related to two other objects that have been much studied: *topological quantum field theories* in 2+1 spacetime dimensions, and *conformal field theories* in 1+1 spacetime dimensions. However, we will just call it an *anyon model*.

9.13 Simulating anyons with a quantum circuit

A *topological quantum computation* is executed in three steps:

1. *Initialization*: Particle-antiparticle pairs $c_1\bar{c}_1, c_2\bar{c}_2, c_3\bar{c}_3, \ldots, c_m\bar{c}_m$ are created. Each pair is of a specified type and has trivial total charge.

2. *Processing*. The $n = 2m$ particles are guided along trajectories, their world lines following a specified braid.

3. *Readout*. Pairs of neighboring particles are fused together, and it is recorded whether each pair annihilates fully or not. This record is the output of the computation.

(In the case of the nonabelian superconductor model of computation, we allowed the braiding to be conditioned on the outcome of fusing carried out during the processing stage. But now we are considering a model in which all measurements are delayed until the final readout.)

How powerful is this model of computation? I claim that this topological quantum computer can be simulated efficiently by a quantum circuit. Since the topological Hilbert space of n anyons does not have a simple and natural decomposition as a tensor product of small subsystems, this claim may not be immediately obvious. To show it we must explain:

1. How to encode the topological Hilbert space using ordinary qubits.

2. How to represent braiding efficiently using quantum gates.

3. How to simulate the fusion of an anyon pair.

Encoding. Since each pair produced during initialization has trivial total charge, the initial state of the n anyons also has trivial total charge. Therefore, the topological Hilbert space is

$$V^1_{a_1 a_2 a_3 \cdots a_n} \cong \bigoplus_{b_1, b_2, \ldots, b_{n-3}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes \cdots \otimes V^{\bar{a}_n}_{b_{n-3} a_{n-1}} , \qquad (9.90)$$

for some choice of the labels $a_1, a_2, a_3, \ldots a_n$; there are $n-3$ intermediate charges and $n-2$ fusion spaces appearing in each summand. Exchanges of the particles swap the labels, but after each exchange the vector space still has the form eq. (9.90) with labels given by some permutation of the original labels.

Although each n-anyon topological Hilbert spaces is not itself a tensor products of subsystems, all of these spaces are contained in

$$(\mathcal{H}_d)^{\otimes(n-2)} , \qquad (9.91)$$

where

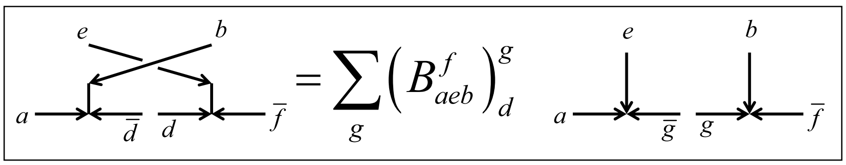
$$\mathcal{H}_d = \bigoplus_{a,b,c} V^1_{abc} . \qquad (9.92)$$

Here, a, b, c are summed over the complete label set of the model (which we have assumed is finite), so that $\mathcal{H}_d$ contains all the possible fusion states of three particles, and the dimension d of $\mathcal{H}_d$ is

$$d = \sum_{a,b,c} N^1_{abc} . \qquad (9.93)$$

Thus the state of n anyons can be encoded in the Hilbert space of $n-2$ *qudits* for some constant d (which depends on the anyon model but is independent of n). The basis states of this qudit can be chosen to be $\{|a,b,c;\mu\rangle\}$, where μ labels an element of the basis for the fusion space V^1_{abc}.

Braiding. In the topological quantum computer, a braid is executed by performing a sequence of exchanges, each acting on a pair of neighboring particles. The effect of each exchange in the standard basis is described by the B-matrix. How is B represented acting on our encoding of the topological vector space (using qudits)? Suppressing fusion states, our basis for two-qudit states can be denoted $|a,b,c\rangle|d,e,\bar{f}\rangle$. But in the topological quantum computer, the labels d and $\bar{c}$ always match, and therefore to perform our simulation of braiding we need only consider two-qudit states whose labels match in this sense:

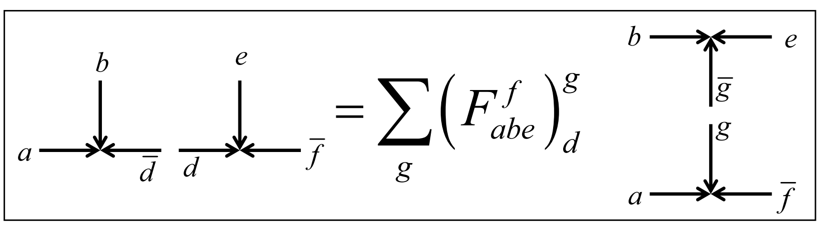


Then the action of the B-matrix on these basis states is

$$B: |a,b,\bar{d}\rangle|d,e,\bar{f}\rangle \mapsto \sum_g |a,e,\bar{g}\rangle|g,b,\bar{f}\rangle \left(B^f_{aeb}\right)^g_d \ . \tag{9.94}$$

As desired, we have represented the B as a $d^2 \times d^2$ matrix acting on a pair of neighboring qudits.

Fusion. Fusion of a pair of anyons can be simulated by a two-qudit measurement, which can be reduced to a single-qudit measurement with a little help from the F-matrix:



Consider a basis state $|a,b,\bar{d}\rangle|d,e,\bar{f}\rangle$ for a pair of neighboring qudits; what is the amplitude for the anyon pair (be) to have trivial total charge? Using an F-move, the state can be expanded as

$$\begin{aligned} F: \qquad & |a,b,\bar{d}\rangle|d,e,\bar{f}\rangle \mapsto \sum_g |a,g,\bar{f}\rangle|b,\bar{g},e\rangle \left(F^f_{abe}\right)^g_d \\ = \quad & |a,1,\bar{f}\rangle|b,1,e\rangle \left(F^f_{abe}\right)^1_d + \sum_{g\neq 1} |a,g,\bar{f}\rangle|b,\bar{g},e\rangle \left(F^f_{abe}\right)^g_d \ ; \end{aligned} \tag{9.95}$$

we have separated the sum over g into the component for which (be) fuses to 1, plus the remainder. After the F-move which (is just a particular two-qudit unitary gate), we can sample the probability that (be) fuses to 1 by performing a projective measurement of the second qudit in the basis $\{|b, \bar{g}, e\rangle\}$, and recording whether $g = 1$.

This completes our demonstration that a quantum circuit can simulate efficiently a topological quantum computer.

9.14 Fibonacci anyons

Now we have established that topological quantum computation is no more powerful than the quantum circuit model — any problem that can be solved efficiently by braiding nonabelian anyons can also be solved efficiently with a quantum circuit. But is it *as* powerful? Can we simulate a universal quantum computer by braiding anyons? The answer depends on the specific properties of the anyons: some nonabelian anyon models are universal, others are not. To find the answer for a particular anyon model, we need to understand the properties of the representations of the braid group that are determined by the F-matrix and R-matrix.

Rather than give a general discussion, we will study one especially simple nonabelian anyon model, and demonstrate its computational universality. This model is the very simplest nonabelian model — conformal field theorists call it the "Yang-Lee model," but I will call it the "Fibonacci model" for reasons that will soon be clear.

In the Fibonacci model there are only two labels — the trivial label, which I will now denote 0, and a single nontrivial label that I will call 1, where $\bar{1} = 1$. And there is only one nontrivial fusion rule:

$$1 \times 1 = 0 + 1 \; ; \tag{9.96}$$

when two anyons are brought together they either annihilate, or fuse to become a single anyon. The model is nonabelian because two anyons can fuse in two distinguishable ways.

Consider the standard basis for the Hilbert space $V_{1^n}^b$ of n anyons, where each basis element describes a distinguishable way in which the n anyons could fuse to give total charge $b \in \{0, 1\}$. If the two anyons furthest to the left were fused first, the resulting charge could be 0 or 1; this charge could then fuse with the third anyon, yielding a total charge of 0 or 1, and so on. Finally, the last anyon fuses with the total charge of the first $n-1$ anyons to give the total charge b. Altogether $n-2$ intermediate charges $b_1, b_2, b_3, \ldots b_{n-2}$ appear in this description of the fusion process; thus the corresponding basis element can be designated with a binary string of length $n-2$. If the total charge is 0, the result of fusing the first $n-1$ anyons *has* to be 1, so the basis states are labeled by strings of length $n-3$.

However, not *all* binary strings are allowed — a 0 must always be followed by a 1. There cannot be two zeros in a row because when the charge 0 fuses with 1, a total charge of 1 is the only possible outcome. Otherwise, there is no restriction on the sequence. Therefore, the basis states are in one-to-one with the binary strings that do not contain two successive 0's.

Thus the dimensions $N_n^0 \equiv N_{1^n}^0$ of the topological Hilbert spaces $V_{1^n}^0$ obey a simple recursion relation. If the fusion of the first two particles yields trivial total charge, then the remaining $n-2$ particles can fuse in N_{n-2}^0 distinguishable ways, and if the fusion of the first two particles yields an anyon with nontrivial charge, then that anyon can fuse with the other $n-2$ anyons in N_{n-1}^0 ways; therefore,

$$N_n^0 = N_{n-1}^0 + N_{n-2}^0 \ . \tag{9.97}$$

Since $N_1^0 = 0$ and $N_2^0 = 1$, the solution to this recursion relation is

$$\begin{array}{rcccccccccc} n & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\ldots \\ N_n^0 & = & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21\ldots \end{array} \tag{9.98}$$

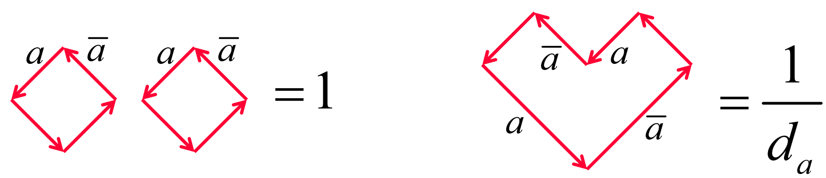
— the dimensions are Fibonacci numbers (which is why I am calling this model the "Fibonacci model").

The Fibonacci numbers grow with n at a rate $N_n^0 \approx C\phi^n$, where ϕ is the golden mean $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.618$. Because ϕ governs the rate at which the Hilbert space enlarges as anyons are added, we say that $d = \phi$ is the *quantum dimension* of the Fibonacci anyon. That this "dimension" is an irrational number illustrates vividly that the topological Hilbert space has no natural decomposition as a tensor product of subsystems — instead, the topologically encoded quantum information is a collective property of the n anyons.

9.15 Quantum dimension

We will return shortly to the properties of the Fibonacci model, but first let's explore more deeply the concept of quantum dimension. For a general anyon model, how should the dimension d_a of label a be defined? For this purpose, it is convenient to imagine a physical process in which two $a\bar{a}$ pairs are created (each with trivial total charge); then the particle a from the pair on the right fuses with the antiparticle $\bar{a}$ from the pair on the left. Do these particles annihilate?

With suitable phase conventions, the *amplitude* for the annihilation to occur is a real number in the unit interval [0,1]. Let us define this number to be $1/d_a$, where d_a is the quantum dimension of a (and $1/d_a^2$ is the *probability* that annihilation occurs). Note that it is clear from this definition that $d_a = d_{\bar a}$. For the case in which the a is the label of an irreducible representation R_a of a group G, the dimension is just $d_a = |R_a|$, the dimension of the representation. This is easily understood pictorially:



If two pairs are created and then each pair annihilates immediately, the world lines of the pairs form two closed loops, and $|R|$ counts the number of distinct "colors" that propagate around each loop. But if the particle from each pair annihilates the antiparticle from the other pair, there is only one closed loop and therefore one sum over colors; if we normalize the process on the left to unity, the amplitude for the process on the right is suppressed by a factor of $1/|R|$. To say the same thing in an equation, the normalized state of an $R\bar R$ pair is

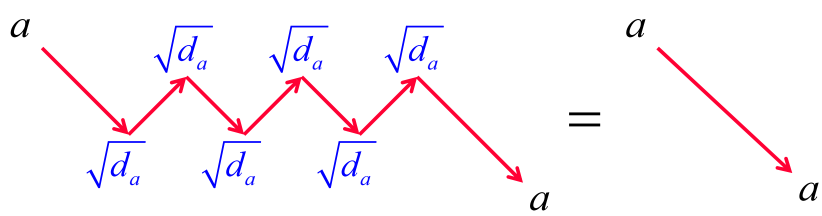
$$|R\bar R\rangle = \frac{1}{\sqrt{|R|}}\sum_i |i\rangle|\bar i\rangle \ , \tag{9.99}$$

where $\{|i\rangle\}$ denotes an orthonormal basis for R and $\{|\bar i\rangle\}$ is a basis for $\bar R$. Suppose that two pairs $|R\bar R\rangle$and $|R'\bar R'\rangle$ are created; if the pairs are fused after swapping partners, the amplitude for annihilation is

$$\begin{aligned}\langle R\bar R, R'\bar R'|R\bar R', R'\bar R\rangle &= \frac{1}{|R|^2}\sum_{i,i',j,j'}\langle j\bar j, j'\bar j'|i\bar i', i'\bar i\rangle \\ &= \frac{1}{|R|^2}\sum_{i,i',j,j'}\delta_{ji}\delta_{ji'}\delta_{j'i'}\delta_{j'i} = \frac{1}{|R|^2}\sum_i \delta_{ii} = \frac{1}{|R|} \ .\end{aligned} \tag{9.100}$$

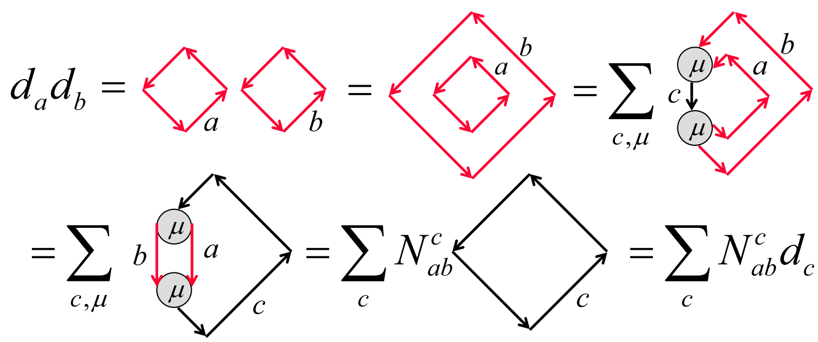
In general, though, the quantum dimension has no direct interpretation in terms of counting "colors," and there is no reason why it has to be an integer.

How are such quantum dimensions related to the dimensions of topological Hilbert spaces? To see the connection, if is very useful to alter our normalization conventions. Notice we can introduce many "zigzags" in the world line of a particle of type a by creating many $a\bar a$ pairs, and fusing the particle from each pair with the antiparticle from the neighboring pair. However, each zigzag reduces the amplitude by another factor of $1/d_a$. We can compensate for these factors of $1/d_a$ if we weight each pair creation or annihilation event by a factor of $\sqrt{d_a}$. With this new convention, we can bend the world line of a particle forward or backward in time without paying any penalty:



Now the weight assigned to a world line is a topological invariant (it is unchanged when we distort the line), and a world line of type a forming a closed loop is weighted by d_a.

With our new conventions, we can justify this sequence of manipulations:



Each diagram represents an inner product of two (unconventionally normalized) states. We have inserted a complete sum over the labels (c) and the corresponding fusion states (μ) that can arise when a and b fuse. Exploiting the topological invariance of the diagram, we have then turned it "inside out," then contracted the fusion states (acquiring the factor N_{ab}^c which counts the possible values of μ).

The equation that we have derived,

$$d_a d_b = \sum_c N_{ab}^c d_c \equiv \sum_c (N_a)_b^c\, d_c\ , \tag{9.101}$$

says that the vector $\vec{d}$, whose components are the quantum dimensions, is an eigenvector with eigenvalue d_a of the matrix N_a that describes how the label a fuses with other labels:

$$N_a \vec{d} = d_a \vec{d} \ . \tag{9.102}$$

Furthermore, since N_a has nonnegative entries and all components of $\vec{d}$ are positive, d_a is the largest eigenvalue of N_a and is nondegenerate. (This simple observation is sometimes called the *Perron-Frobenius theorem.*) For n anyons, each with label a, the topological Hilbert space $V^b_{aaa\cdots a}$ for the sector with total charge b has dimension

$$N^b_{aaa\cdots a} = \sum_{\{b_i\}} N^{b_1}_{aa} N^{b_2}_{ab_1} N^{b_3}_{ab_2} \dots N^{b}_{ab_{n-2}} = \langle b| \left(N_a\right)^{n-1} |a\rangle \ . \tag{9.103}$$

The matrix N_a can be diagonalized, and expressed as

$$N_a = |v\rangle d_a \langle v| + \cdots \ , \tag{9.104}$$

where

$$|v\rangle = \frac{\vec{d}}{\mathcal{D}} \ , \quad \mathcal{D} = \sqrt{\sum_c {d_c}^2} \ , \tag{9.105}$$

and subleading eigenvalues have been omitted; therefore

$$N^b_{aaa\cdots a} = d_a^n d_b / \mathcal{D}^2 + \cdots \ , \tag{9.106}$$

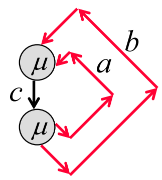
where the ellipsis represents terms that are exponentially suppressed for large n. We see that the quantum dimension d_a controls the rate of growth of the n-particle Hilbert space for anyons of type a.

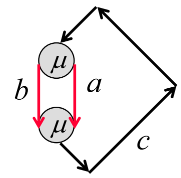
Because the label 0 with trivial charge fuses trivially, we have $d_0 = 1$. In the case of the Fibonacci model, it follows from the fusion rule $1\times 1 = 0+1$ that $d_1^2 = 1 + d_1$, which is solved by $d_1 = \phi$ as we found earlier; therefore $\mathcal{D}^2 = d_0^2 + d_1^2 = 1 + \phi^2 = 2 + \phi$. Our formula becomes

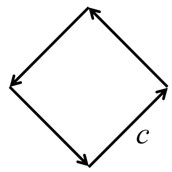
$$N^0_{111\cdots 1} = \left(\frac{1}{2+\phi}\right) \phi^n \ , \tag{9.107}$$

which is an excellent approximation to the Fibonacci numbers even for modest values of n.

Suppose that an $a\bar{a}$ pair and a $b\bar{b}$ pair are both created. If the a and b particles are fused, with what probability $p(ab \to c)$ will their total charge be c? This question can be answered using the same kind of graphical manipulations:

$$d_a d_b\, p(ab \to c) = \sum_{\mu}$$


$$= \sum_{\mu}$$


$$= N^c_{ab}$$


$$= N^c_{ab} d_c$$

Dividing by $d_a d_b$ to restore the proper renormalization of the inner product, we conclude that

$$p(ab \to c) = \frac{N^c_{ab} d_c}{d_a d_b} \ , \tag{9.108}$$

which generalizes the formula $p(a\bar{a} \to 1) = 1/d_a^2$ that we used to define the quantum dimension, and satisfies the normalization condition

$$\sum_c p(ab \to c) = 1. \tag{9.109}$$

To arrive at another interpretation of the quantum dimension, imagine that a dense gas of anyons is created, which is then permitted to anneal for awhile — anyons collide and fuse, gradually reducing the population of particles. Eventually, but long before the thermal equilibrium is attained, the collision rate becomes so slow that the fusion process effectively turns off. By this stage, whatever the initial distribution of particles types, a steady state distribution is attained that is preserved by collisions. If in the steady state particles of type a appear with probability p_a, then

$$\sum_{ab} p_a p_b \ p(ab \to c) = p_c \ . \tag{9.110}$$

Using

$$\sum_a N^c_{ab} \ d_a = \sum_a N^a_{b\bar{c}} \ d_{\bar{a}} = d_b d_{\bar{c}} = d_b d_c \ , \tag{9.111}$$

we can easily verify that this condition is satisfied by

$$p_a = \frac{d_a^2}{\mathcal{D}^2} \ . \tag{9.112}$$

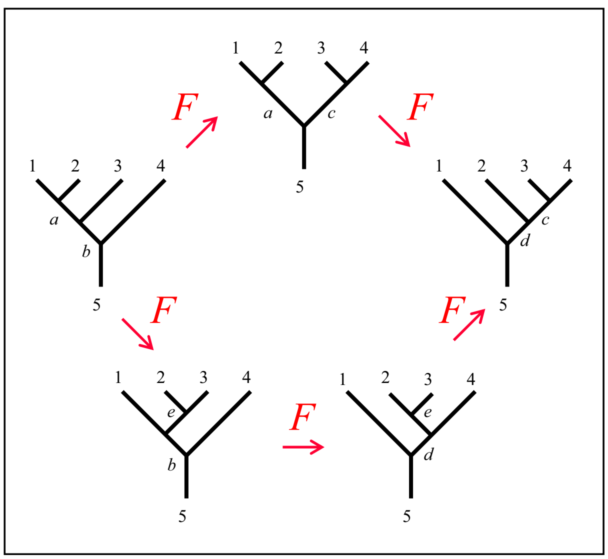
We conclude that if anyons are created in a random process, those carrying labels with larger quantum dimension are more likely to be produced, in keeping with the property that anyons with larger dimension have more quantum states.

9.16 Pentagon and hexagon equations

To assess the computational power of an anyon model like the Fibonacci model, we need to know the braiding properties of the anyons, which are determined by the R and F matrices. We will see that the braiding rules are highly constrained by algebraic consistency conditions. For the Fibonacci model, these consistency conditions suffice to determine a unique braiding rule that is compatible with the fusion rules.

Consistency conditions arise because we can make a sequence of "F-moves" and "R-moves" to obtain an isomorphism relating two topological Hilbert spaces. The isomorphism can be regarded as a unitary matrix that relates the canonical orthonormal bases for two different spaces; this unitary transformation does not depend on the particular sequence of moves from which the isomorphism is constructed, only on the initial and final bases.

For example, there are five different ways to fuse four particles (without any particle exchanges), which are related by F-moves:



The basis shown furthest to the left in this pentagon diagram is the "left standard basis" $\{|\text{left}; a, b\rangle\}$, in which particles 1 and 2 are fused first, the resulting charge a is fused with particle 3 to yield charge b, and then finally b is fused with particle 4 to yield the total charge 5. The basis shown furthest to the right is the "right standard basis" $\{|\text{right}; c, d\rangle\}$, in which the particles are fused from right to left instead of left to right. Across the top of the pentagon, these two bases are related by two F-

moves, and we obtain

$$|\text{left}; a, b\rangle = \sum_{c,d} |\text{right}; c, d\rangle \left(F^5_{12c}\right)^d_a \left(F^5_{a34}\right)^c_b \ . \tag{9.113}$$

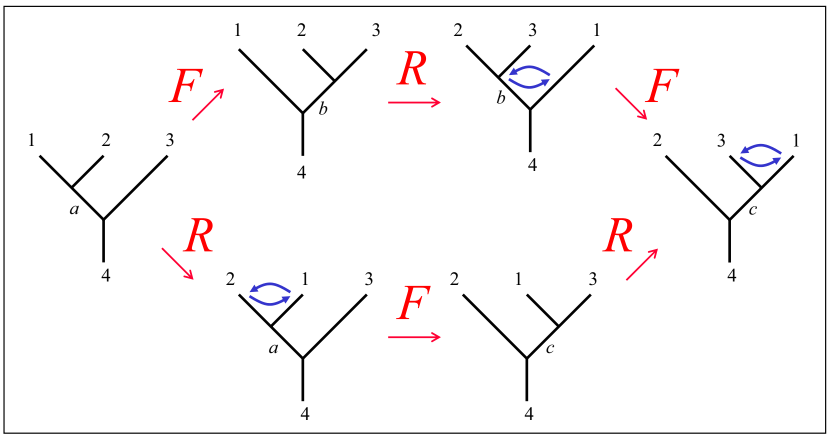
Across the bottom of the pentagon, the bases are related by three F-moves, and we find

$$|\text{left}; a, b\rangle = \sum_{c,d,e} |\text{right}; c, d\rangle \left(F^d_{234}\right)^c_e \left(F^5_{1e4}\right)^d_b \left(F^b_{123}\right)^e_a \ . \tag{9.114}$$

Equating our two expressions for $|\text{left}; a, b\rangle$, we obtain the *pentagon equation*:

$$\left(F^5_{12c}\right)^d_a \left(F^5_{a34}\right)^c_b = \left(F^d_{234}\right)^c_e \left(F^5_{1e4}\right)^d_b \left(F^b_{123}\right)^e_a \ . \tag{9.115}$$

Another nontrivial consistency condition is found by considering the various ways that three particles can fuse:



The basis $\{|\text{left}; a\rangle\}$ furthest to the left in this hexagon diagram is obtained if the particles are arranged in the order 123, and particles 1 and 2 are fused first, while the basis $\{|\text{right}, c\rangle\}$ furthest to the right is obtained if the particles are arranged in order 231, and particles 1 and 3 are fused first. Across the top of the hexagon, the two bases are related by the sequence of moves FRF:

$$|\text{left}, a\rangle = \sum_{b,c} |\text{right}; c\rangle \left(F^4_{231}\right)^c_b R^4_{1b} \left(F^4_{123}\right)^b_a \ . \tag{9.116}$$

Across the bottom of the hexagon, the bases are related by the sequence of moves RFR, and we find

$$|\text{left}, a\rangle = \sum_{c} |\text{right}; c\rangle R^c_{13} \left(F^4_{213}\right)^c_a R^a_{12} \ . \tag{9.117}$$

Equating our two expressions for $|\text{left}; a\rangle$, we obtain the *hexagon equation*:

$$R_{13}^{c}\left(F_{213}^{4}\right)_{a}^{c} R_{12}^{a}=\sum_{b}\left(F_{231}^{4}\right)_{b}^{c} R_{1b}^{4}\left(F_{123}^{4}\right)_{a}^{b} . \tag{9.118}$$

A beautiful theorem, which I will not prove here, says that there are no further conditions that must be imposed to ensure the consistency of braiding and fusing. That is, for any choice of an initial and final basis for n anyons, *all* sequences of R-moves and F-moves that take the initial basis to the final basis yield the *same* isomorphism, provided that the pentagon equation and hexagon equation are satisfied. This theorem is an instance of the *MacLane coherence theorem*, a fundamental result in category theory. The pentagon and hexagon equations together are called the *Moore-Seiberg polynomial equations* — their relevance to physics was first appreciated in studies of (1+1)-dimensional conformal field theory during the 1980's.

A solution to the polynomial equations defines a viable anyon model. Therefore, there is a systematic procedure for constructing anyon models:

1. Choose a set of labels and assume a fusion rule.

2. Solve the polynomial equations for R and F.

If no solutions exist, then the hypothetical fusion rule is incompatible with the principles of local quantum physics and must be rejected. If there is more than one solution (not related to one another by any reshuffling of the labels, redefinition of bases, etc.), then each distinct solution defines a distinct model with the assumed fusion rule.

To illustrate the procedure, consider the polynomial equations for the Fibonacci fusion rule. There are only two F-matrices that arise, which we will denote as

$$F_{0111} \equiv F_0 , \quad F_{1111} \equiv F_1 . \tag{9.119}$$

F_0 is really the 1×1 matrix

$$(F_0)_a^b = \delta_a^1 \delta_1^b , \tag{9.120}$$

while F_1 is a 2×2 matrix. The pentagon equation becomes

$$(F_c)_a^d (F_a)_b^c = \sum_e (F_d)_e^c (F_e)_b^d (F_b)_a^c . \tag{9.121}$$

The general solution for $F \equiv F_1$ is

$$F = \begin{pmatrix} \tau & e^{i\phi}\sqrt{\tau} \\ e^{-i\phi}\sqrt{\tau} & -\tau \end{pmatrix} , \tag{9.122}$$

where $e^{i\phi}$ is an arbitrary phase (which we can set to 1 with a suitable phase convention), and $\tau = \left(\sqrt{5}-1\right)/2 = \phi - 1 \approx .618$, which satisfies

$$\tau^2 + \tau = 1 \ . \tag{9.123}$$

The 2×2 R-matrix that describes a counterclockwise exchange of two Fibonacci anyons has two eigenvalues — R^0 for the case where the total charge of the pair of anyons is trivial, and R^1 for the case where the total charge is nontrivial. The hexagon equation becomes

$$R^c \left(F\right)^c_a R^a = \left(F\right)^c_0 \left(F\right)^0_a + \left(F\right)^c_1 R^1 \left(F\right)^1_a \ . \tag{9.124}$$

Using the expression for F found by solving the pentagon equation, we can solve the hexagon equation for R, finding

$$R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix} \ , \quad F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix} \ . \tag{9.125}$$

The only other solution is the complex conjugate of this one; this second solution really describes the same model, but with clockwise and counterclockwise braiding interchanged. Therefore, an anyon model with the Fibonacci fusion rule really *does* exist, and it is essentially unique.

9.17 Simulating a quantum circuit with Fibonacci anyons

Now we know enough to address whether a universal quantum computer can be simulated using Fibonacci anyons. We need to explain how qubits can be encoded with anyons, and how a universal set of quantum gates can be realized.

First we note that the Hilbert space $V_4^0 \equiv V_{1111}^0$ has dimension $N_4^0 = 2$; therefore a qubit can be encoded by four anyons with trivial total charge. The anyons are lined up in order 1234, numbered from left to right; in the standard basis state $|0\rangle$, anyons number 1 and number 2 fuse to yield total charge 0, while in the standard basis state $|1\rangle$, anyons 1 and 2 fuse to yield total charge 1. Acting on this standard basis, the braid group generator σ_1 (counterclockwise exchange of particles 1 and 2) is represented by

$$\sigma_1 \mapsto R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix} \ , \tag{9.126}$$

while the generator σ_2 is represented by

$$\sigma_2 \mapsto B = F^{-1} R F \ , \quad F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix} \ . \tag{9.127}$$

These matrices generate a representation of the braid group B_3 on three strands whose image is dense in SU(2). Indeed, R and B generate Z_{10} subgroups of U(2), about two distinct axes, and there is no finite subgroup of U(2) that contains both of these subgroups — therefore, the representation closes on the group containing all elements of U(2) with determinant equal to a 10th root of unity. Similarly, for n anyons with trivial total charge, the image of the representation of the braid group is dense in $\mathrm{SU}(N_n^0)$.

To simulate a quantum circuit acting on n qubits, altogether $4n$ anyons are used. We have just seen that by braiding within each cluster of four anyons, arbitrary single-qubit gates can be realized. To complete a universal set, we will need two-qubit gates as well. But two neighboring qubits are encoded by eight anyons, and exchanges of these anyons generate a representation of B_8 whose image is dense in $\mathrm{SU}(N_8^0)= \mathrm{SU}(13)$, which of course includes the SU(4) that acts on the two encoded qubits. Therefore, each gate in a universal set can be simulated with arbitrary accuracy by some finite braid.

Since we can braid clockwise as well as counterclockwise, the inverse of each exchange gate is also in our repertoire. Therefore, we can apply the Solovay-Kitaev theorem to conclude that the universal gates of the circuit model can be simulated to accuracy ε with braids of length $\mathrm{poly}\,(\log(1/\varepsilon))$. It follows that an ideal quantum circuit with L gates acting on all together n qubits can be simulated to fixed accuracy using $4n$ anyons and a braid of length $O(L \cdot \mathrm{poly}(\log(L))$. As desired, we have shown that a universal quantum computer can be simulated efficiently with Fibonacci anyons. Note that, in contrast to the simulation using the nonabelian superconductor model, no intermediate measurements are needed to realize the universal gates.

In the analysis above, we have assumed that there are no errors in the simulation other than those limiting the accuracy of the Solovay-Kitaev approximation to the ideal gates. It is therefore implicit that the temperature is small enough compared to the energy gap of the model that thermally excited anyons are too rare to cause trouble, that the anyons are kept far enough apart from one another that uncontrolled exchange of charge can be neglected, and in general that errors in the topological quantum computation are unimportant. If the error rate is small but not completely negligible, then the standard theory of quantum fault tolerance can be invoked to boost the accuracy of the simulation as needed, at an additional overhead cost polylogarithmic in L. The fault-tolerant procedure should include a method for controlling the "leakage" of the encoded qubits — that is, to prevent the drift of the clusters of four qubits from the two-dimensional computational space V_4^0 to its three-dimensional orthogonal complement V_4^1.

9.18 Epilogue

That is as far as I got in class. I will mention briefly here a few other topics that I might have covered if I had not run out of time.

9.18.1 Chern-Simons theory

We have discussed how anyon models can be constructed through a brute-force solution to the polynomial equations. This method is foolproof, but in practice models are often constructed using other, more efficient methods. Indeed, most of the known anyon models have been found as instances of *Chern-Simons theory*.

The fusion rules of a Chern-Simons theory are a truncated version of the fusion rules for irreducible representations of a Lie group. For example, associated with the group SU(2) there is a tower of Chern-Simons theories indexed by a positive integer k. For SU(2), the irreducible representations carry labels $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$, and the fusion rules have the form

$$j_1 \times j_2 = \sum_{j=|j_2-j_1|}^{j_1+j_2} j \ . \tag{9.128}$$

In the Chern-Simons theory denoted $\mathrm{SU}(2)_k$, the half-integer labels are limited to $j \le k/2$, and the label j is contained in $j_1 \times j_2$ only if $j_1 + j_2 + j \le k$.

For example, the $\mathrm{SU}(2)_1$ model is abelian, and the nontrivial fusion rules of the $\mathrm{SU}(2)_2$ model are

$$\begin{aligned} \tfrac{1}{2} \times \tfrac{1}{2} &= 0 + 1 \ , \\ \tfrac{1}{2} \times 1 &= \frac{1}{2} \ , \\ 1 \times 1 &= 0 \ . \end{aligned} \tag{9.129}$$

Therefore, the label $\frac{1}{2}$ has quantum dimension $d_{1/2} = \sqrt{2}$, and the topological Hilbert space of $2m$ such anyons with total charge 0 has dimension

$$N^0_{(\frac{1}{2})^{2m}} = 2^{m-1} \ . \tag{9.130}$$

The polynomial equations for these fusion rules have multiple solutions (only one of which describes the braiding properties of the $\mathrm{SU}(2)_2$ model), but none of the resulting models have computationally universal braiding rules. The space $V^0_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}$ is two-dimensional, and the 2×2 matrices $F \equiv F_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}$ and $R \equiv R_{\frac{1}{2}\frac{1}{2}}$ are, up to overall phases and complex conjugation,

$$F = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ , \quad R = P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \ . \tag{9.131}$$

There are Clifford-group quantum gates, inadequate for universality.

However, the $\mathrm{SU}(2)_k$ models for $k \geq 3$ *are* computationally universal. The nontrivial fusion rules of $\mathrm{SU}(2)_3$ are

$$\begin{aligned}
\tfrac{1}{2} \times \tfrac{1}{2} &= 0 + 1 \ , \\
\tfrac{1}{2} \times 1 &= \frac{1}{2} + \frac{3}{2} \ , \\
\tfrac{1}{2} \times \tfrac{3}{2} &= 1 \ , \\
1 \times 1 &= 0 \ + 1 \ , \\
1 \times \tfrac{3}{2} &= \frac{1}{2} \ , \\
\tfrac{3}{2} \times \tfrac{3}{2} &= 0 \ .
\end{aligned} \tag{9.132}$$

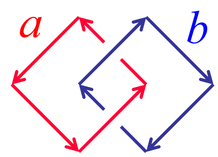
The Fibonacci (Yang-Lee) model that we have studied is obtained by truncating $\mathrm{SU}(2)_3$, further, eliminating the noninteger labels $\frac{1}{2}$ and $\frac{3}{2}$ (*i.e.*, this is the Chern-Simons theory $\mathrm{SO}(3)_3$); then the only remaining nontrivial fusion rule is $1 \times 1 = 0 + 1$.

Wang (unpublished) has recently constructed all anyons models with no more than four labels, and has found that all of the models are closely related to the models that are found in Chern-Simons theory.

9.18.2 S-matrix

The *modular S-matrix* of an anyon model can be defined in terms of two anyon world lines that form a *Hopf link*:

$$S_a^b = \frac{1}{\mathcal{D}}$$



Here $\mathcal{D}$ is the total quantum dimension of the model, and we have used the normalization where unlinked loops would have the value $d_a d_b$; then the matrix S_a^b is symmetric and unitary. In abelian anyon models, the Hopf link arose in our discussion of topological degeneracy, where we characterized how the vacuum state of an anyon model on the torus is affected when an anyon is transported around one of the cycles of the torus. The S-matrix has a similar interpretation in the nonabelian case. By elementary reasoning, S can be related to the fusion rules:

$$(N_a)_b^c = \sum_d S_b^d \left(\frac{S_a^d}{S_1^d} \right) \left(S^{-1}\right)_d^c \ ; \tag{9.133}$$

that is, the S-matrix simulaneously *diagonalizes* all the matrices $\{N_a\}$ (the *Verlinde relation*). Note that it follows from the definition that $S_1^a = d_a/\mathcal{D}$.

9.18.3 Edge excitations

In our formulation of anyon models, we have discussed the fusing and braiding of particles in the two-dimensional *bulk*. But there is another aspect of the physics of two-dimensional media that we have not yet discussed, the properties of the one-dimensional *edge* of the sample. Typically, if a two-dimensional system supports anyons in the bulk, there are also *chiral massless excitations* that propagate along the one-dimensional edge. At nonzero temperature T, there is an energy flux along the edge given by the expression

$$J = \frac{\pi}{12} c_- T^2 ; \tag{9.134}$$

here the constant c_-, called the *chiral central charge* of the edge, is a universal property that is unaffected by small changes in the underlying Hamiltonian of the system.

While this chiral central charge is an intrinsic property of the two-dimensional medium, the properties of the anyons in the bulk do not determine it completely; rather we have

$$\frac{1}{\mathcal{D}} \sum_a d_a^2 e^{2\pi i J_a} = e^{(2\pi i/8)c_-} , \tag{9.135}$$

where the sum is over the complete label set of the anyon model, and $e^{2\pi i J_a} = R^1_{a\bar{a}}$ is the topological spin of the label a. This expression relates the quantity c_-, characteristic of the edge theory, to the quantum dimensions and topological spins of the bulk theory, but determines c_- only modulo 8. Therefore, at least in principle, there can be multiple edge theories corresponding to a single theory of anyons in the bulk.

9.19 Bibliographical notes

Some of the pioneering papers on the theory of anyons are reprinted in [1].

What I have called the "nonabelian superconductor" model is often referred to in the literature as the "quantum double," and is studied using the representation theory of Hopf algebras. For a review see [2].

That nonabelian anyons can be used for fault-tolerant quantum computing was first suggested in [3]. This paper also discusses the toric code, and related lattice models that have nonabelian phases. A particular realization of universal quantum computation in a nonabelian superconductor was discussed in [4, 5]. My discussion of the universal gate set is based on [6], where more general models are also discussed. Other schemes, that make more extensive use of electric charges and that are universal for smaller groups (like S_3) are described in [7].

Diagrammatic methods, like those I used in the discussion of the quantum dimension, are extensively applied to derive properties of anyons in [8]. The role of the polynomial equations (pentagon and hexagon equations) in (1+1)-dimensional conformal field theory is discussed in [9].

Simulation of anyons using a quantum circuit is discussed in [10]. Simulation of a universal quantum computer using the anyons of the $\mathrm{SU}(2)_{k=3}$ Chern-Simons theory is discussed in [11]. That the Yang-Lee model is also universal was pointed out in [12].

I did not discuss physical implementations in my lectures, but I list a few relevant references here anyway: Ideas about realizing abelian and nonabelian anyons using superconducting Josephson-junction arrays are discussed in [13]. A spin model with nearest-neighbor interactions that has nonabelian anyons (though not ones that are computationally universal) is proposed and solved in [14], and a proposal for realizing this model using cold atoms trapped in an optical lattice is described in [15]. Some ideas about realizing the (computationally universal) $\mathrm{SU}(2)_{k=3}$ model in a system of interacting electrons are discussed in [16].

Much of my understanding of the theory of computing with nonabelian anyons was derived from many helpful discussions with Alexei Kitaev.

References

[1] F. Wilczek, *Fractional statistics and anyon superconductivity* (World Scientific, Singapore, 1990).

[2] M. de Wild Propitius and F. A. Bais, "Discrete gauge theories," arXiv: hep-th/9511201 (1995).

[3] A. Yu. Kitaev, "Fault-tolerant quantum computation by anyons," Annals Phys. **303**, 2-30 (2003), arXiv: quant-ph/9707021.

[4] R. W. Ogburn and H. Preskill, "Topological quantum computation," Lect. Notes in Comp. Sci. **1509**, 341-356 (1999).

[5] J. Preskill, "Fault-tolerant quantum computation," arXiv: quant-ph/9712048 (1997).

[6] C. Mochon, "Anyons from non-solvable finite groups are sufficient for universal quantum computation" Phys. Rev. A **67**, 022315 (2003), quant-ph/0206128.

[7] C. Mochon, "Anyon computers with smaller groups," Phys. Rev. A **69**, 032306 (2004), arXiv: quant-ph/0306063.

[8] J. Fröhlich and F. Gabbiani, "Braid statistics in local quantum theory," Rev. Math. Phys. **2:3**, 251–353 (1990).

[9] G. Moore and N. Seiberg, "Classical and quantum conformal field theory," Comm. Math. Phys. **123**, 171–254 (1989).

[10] M. H. Freedman, A. Kitaev, and Z. Wang, "Simulation of topological field theories by quantum computers," Comm. Math. Phys. **227**, 587-603 (2002), arXiv: quant-ph/0001071.

[11] M. H. Freedman, M. Larsen, and Z. Wang, "A modular functor which is universal for quantum computation," arXiv: quant-ph/0001108 (2000).

[12] G. Kuperberg, unpublished.

[13] B. Doucot, L. B. Ioffe, and J. Vidal, "Discrete non-Abelian gauge theories in two-dimensional lattices and their realizations in Josephson-junction arrays," Phys. Rev. B **69**, 214501 (2004), arXiv: cond-mat/0302104.

[14] A. Kitaev, "Anyons in a spin model on the honeycomb lattice," unpublished.

[15] L.-M. Duan, E. Demler, and M. D. Lukin, "Controlling spin exchange interactions of ultracold atoms in optical lattices," Phys. Rev. Lett. **91**, 090402 (2003), arXiv: cond-mat/0210564.

[16] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Zhenghan Wang, "A class of P, T-invariant topological phases of interacting electrons," cond-mat/0307511 (2003).