Constant depth and the Clifford hierarchy

The Eastin-Knill Theorem tells us that we can't do a universal set of logical gates on a QECC where each gate in the set is executed transversally. Hence we need other tools to achieve universal fault tolerance, like magic state distillation or code switching. But what principles dictate which gates have simple fault-tolerant realizations for a given code?

One useful guideline comes from the Bravyi-König Theorem:

Consider a family of topological stabilizer codes in D spatial dimensions (geometrically local checks and distance increasing without bound). Suppose a geom. local constant depth unitary circuit executes a logical operation (preserves the code space). Then this logical unitary is contained in the Dth level of the Clifford hierarchy.

Constant depth is desirable in this code family because it limits the number of faults and the number errors spread by the gates to a constant, below the growing distance of the code family.

This statement applies to local codes and circuits. But there is also a corresponding statement that holds for high-rate nonlocal codes, obtained from a product of D chain complexes.

What is the Clifford hierarchy? Each level is a discrete set of unitaries, defined recursively up to a phase.

$$\mathcal{P}_0 = \text{id}_y, \quad \mathcal{P}_1 = \text{Pauli} \quad \mathcal{P}_2 = \text{Clifford}$$

$$\mathcal{P}_\ell = \{ U : UPU^{-1} \in \mathcal{P}_{\ell-1} \ \forall P \in \mathcal{P}_1 \}$$

This definition of $\mathcal{P}_\ell$ for $\ell \geq 3$ extends the notion of the Clifford group (unitaries whose action by conjugation takes Pauli to Pauli). $\mathcal{P}_2$ is a group because if U_1, U_2 map $\mathcal{P}_1$ to $\mathcal{P}_1$ then so does $U_1 U_2$. But $\mathcal{P}_\ell$ for $\ell \geq 3$ is not a group.

Examples of $\mathcal{P}_3$ gates are T, CCZ, and Tof. We exploit the $\mathcal{P}_3$ property in fault-tolerance; it tells us, for example, that we can "clean up" a measurement based T gadget with $|T\rangle$ state input using a (fault-tolerant) Clifford gate correction conditioned on measurement outcome.

when we pull a Pauli frame update through a T gate it becomes a Clifford frame update

$$T X |\psi\rangle = (T X T^{-1}) T|\psi\rangle = \frac{1}{\sqrt{2}}(X+Y)\, T|\psi\rangle$$

Clifford

Examples of higher level gates

P_m gates: $\mathrm{diag}(1, e^{i\pi/2^{m-1}})$

$m=1: Z \quad m=2: S \quad m=3: T \quad m=4: \sqrt{T}$

Going to the next higher level ~ taking square root.

Key idea: Cleaning Lemma (for stabilizer codes). Recall - a set of qubits is correctable if erasure of the set can be corrected (size of set < distance of code is sufficient but not necessary for correctability).

Suppose a set R is correctable and P is a logical Pauli. Then there is a stabilizer element S such that

$$P' = PS = I_R \otimes \tilde{P}'_{R^c}$$

P' is unsupported on R (we can clean P on R by applying the code's check operators).

Note that if a logical unitary is supported on correctable set, then it must be the logical identity up to a phase. (Else: we could apply U and erase R, an uncorrectable erasure error.)

From the perspective of the cleaning lemma: Suppose

$$U = U_R \otimes I_{R^c} \quad \text{(supported on } R\text{)}$$

and let $\underline{P}$ be <u>any</u> logical Pauli, which can be cleaned on R

$$\underline{P}' = S\underline{P} = I_R \otimes \tilde{\underline{P}}'_{R^c}$$

Therefore

$$U\underline{P}'U^{-1} = (U_R \otimes I_{R^c})(I_R \otimes \tilde{\underline{P}}'_{R^c})(U_R^{-1} \otimes I_{R^c})$$

$$= (I_R \otimes \tilde{\underline{P}}'_{R^c}) = \underline{P}'$$

But a unitary that commutes with any logical Pauli must be a multiple of the identity

$$U = \text{idy} \times \text{phase}$$

(because Paulis are a complete operator basis).

Using the cleaning lemma, we can show the following for <u>any</u> stabilizer code: Suppose the n qubits in the code block decompose into m nonoverlapping correct-~~able set~~

$$\cancel{\bigcup_{i=1}^{m} R_i \text{ where } R_i \text{ is correctable}}$$

~~Let U be a logical unitary which is transversal w.r.t. this partition~~

$$\cancel{U = \bigotimes_{i=1}^{m} U_i}$$

able sets, plus one more set that need not be correctable

$$\text{code block} = \bigcup_{i=1}^{m} R_i \cup \tilde{R},$$

and let unitary U act transversally on the correctable sets:

$$U = \bigotimes_{i=1}^{m} U_i \otimes I_{\tilde{R}}$$

Then $U \in \mathcal{P}_{m-1}$ (is in level $m-1$ of hierarchy). Here we make no further assumptions about code distance or geometric locality of the checks.

The proof is by induction. The $m=1$ case was described above. Now consider $m=2$: code block is $R_1 \cup R_2 \cup \tilde{R}$ where R_1 and R_2 are both correctable, and $U = U_1 \otimes U_2 \otimes I_{\tilde{R}}$

Then consider any Pauli operator and clean it on R_2, obtaining $\quad \underline{P}' = \underline{P}_1 \otimes I_2 \otimes \tilde{\underline{P}}$

We find

$$U\underline{P}'U^{-1}\underline{P}'^{-1} = (U_1 \otimes U_2 \otimes \underline{1})(\underline{P}_1 \otimes \underline{I} \otimes \tilde{\underline{P}})(U_1^{-1} \otimes U_2^{-1} \otimes \underline{1})(\underline{P}_1^{-1} \otimes \underline{I} \otimes \tilde{\underline{P}}^{-1})$$

$$= U_1 \underline{P}_1 U_1^{-1} \underline{P}_1^{-1} \otimes I_2 \otimes \tilde{I}$$

This is a unitary supported on correctable set R_1. That is the $m=1$ case, for which we know unitary must be a multiple of the identity

$$\Rightarrow \quad U\underline{P}'U^{-1} = \text{phase} \times \underline{P}'$$

Thus U commutes with every logical Pauli operator, up to a phase. The only such unitary is a Pauli; hence $U \in \mathcal{P}_1$.
If we expand U in the Pauli basis

$$U = \sum_a \alpha_a \underline{P}_a, \text{ then } \underline{P}'^{-1} U \underline{P}' = \eta U \quad (\eta = \text{phase})$$

says that $\underline{P}'^{-1} \underline{P}_a \underline{P}' = \eta \underline{P}_a$ for all a such that $\alpha_a \neq 0$.
Either all such $\underline{P}_a$ commute with $\underline{P}'$ or all anticommute. If there are $\underline{P}_a$ and $\underline{P}_b$ that (say) both commute with $\underline{P}'$ then there is another Pauli $\underline{P}''$ that commutes with $\underline{P}_a$ and anticommutes with $\underline{P}_b$; hence $\underline{P}''$ does not commute with U up to a phase. We conclude that only one α_a is nonzero.

At the next level, code is $R_1 \cup R_2 \cup R_3 \cup \tilde{R}$ and $U = U_1 \otimes U_2 \otimes U_3 \otimes I$ and Pauli P after cleaning is $\underline{P}' = \underline{P}_1' \otimes \underline{P}_2' \otimes I \otimes I \implies$

$U \underline{P}' U^{-1} \underline{P}'^{-1}$ is supported on $R_1 \cup R_2$ and hence is contained in $\mathcal{P}_1 \implies$

$$U \underline{P}' U^{-1} = \underline{P}' \times \text{Pauli} = \text{Pauli}$$

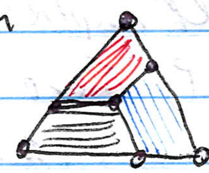
This means $U \in \mathcal{P}_2$

Likewise, after cleaning the Pauli on one of the m regions

$$UP'U^{-1} \text{ supported on } m-1 \text{ regions}$$

and hence is P_{m-1}

Example: the [[7,1,3]] Steane code (a color code). For each 2-cell, checks include XXXX and ZZZZ acting on the four vertices.



vertices = qubits

2-cells = checks

Faces are 3-colorable and vertices are trivalent ⇒ Faces meet on edges (or not at all)

We can choose logical X and Z to be supported on e.g. the 3 bottom vertices (commutes with all checks). This means that top 4 vertices must be a correctable set. No possible nontrivial logical operator is supported there, even though the size of the set (4) is larger than distance ($d=3$). We can decompose the 7 qubits into correctable sets

$$4+2+1 \qquad (2 \text{ and } 1 \text{ are } < d.)$$

Therefore, transversal ~~unitary~~ unitary must be Clifford. (And in fact the entire Clifford group is transversal.)

In 3D [[15, 1, 3]] color code, lattice is 4-valent and 3-cells are 4-colorable. X checks on 3-cells and Z checks on faces. Now decomposition into correctable regions is

$$8+4+2+1 \Rightarrow \text{transversal gates are } P_3$$

In fact, the code admits a transversal T (and then Eastin-Knill implies some Clifford gate is not transversal; there is not a transversal H. In fact Z logicals are strings and X logicals are membranes).

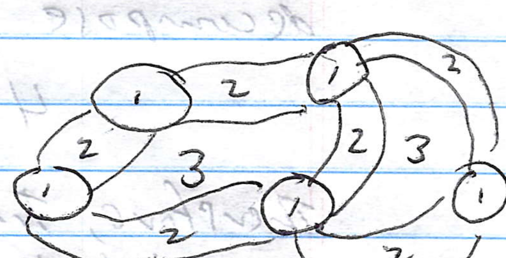
Higher dimension Reed-Muller codes are $[[2^m-1, 1, 3]]$. Decomposition into m regions is

$$2^{m-1}+2^{m-2}+\dots+2+1$$

$\Rightarrow$ ~~[illegible]~~ P_{m-1} gates can be transversal, like $\mathrm{diag}(1, e^{i\pi/2^{m-2}})$

Now let's consider local codes with growing distance, e.g. in $D=2$ dimensions. We triangulate the plane (or a non-Euclidean surface)

And then "fatten" the vertices, edges, and faces



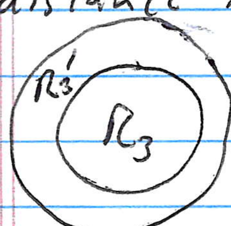
The checks have finite range and the distance is large. Choose connected components of regions 1, 2, 3 to be < distance (hence correctable) and separation between components > range.

Recall the lemma: union of separated correctable sets is correctable. We have decomposed the surface into 3 correctable sets. Hence unitary transversal w.r.t. this decomposition must be in $\mathcal{P}_2$

Similarly, in D dimensions, fattened triangulation decomposes space into D+1 correctable regions. Therefore transversal unitary in $\mathcal{P}_D$.

How does the argument extend to constant depth unitary circuits? Consider D=2 case.

If gates have range r and the circuit for U has depth h, then support of UPU^{-1} reaches beyond the support of P by $\leq rh$ in all directions. Suppose <u>P</u> is <u>Pauli</u>. If the distance is large enough, the connected components



of R_3 can be chosen to be sufficiently separated that regions extending beyond R_3 by $> rh$ are still correctable and separated, so we can clean P on the larger region R_3', ensuring that $UP'U^{-1}$ is unsupported on R_3. Hence $UP'U^{-1}P'^{-1}$ is supported on the union of two correctable sets $R_1 \cup R_2$. This is what we need for the inductive step.

The argument by Bravyi + Konig is a little more sophisticated, and applies even in the case of a code deformation — where U does not preserve the code space but instead maps it to a different topological stabilizer code.