Ph/CS 219C

Exercises
Due: Thursday 7 May 2026

2.1 A three-dimensional hypergraph product code

As discussed in class (and in the notes), a classical code with parity check matrix A is associated with the chain complex

$$C_A : 0 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0. \tag{1}$$

If A is an $r_A \times n_A$ matrix, then A_1 is an n_A-dimensional vector space, A_0 is an r_A-dimensional vector space, ∂_1 is the parity check H, ∂_0 has a trivial image, and ∂_2 has a trivial kernel. The homology group

$$H_1(C_A) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \ker A \tag{2}$$

is the code space, and the homology group

$$H_0(C_A) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \ker A^T \tag{3}$$

is the transpose code with parity check matrix A^T.

We saw that a "two-dimensional" hypergraph product code can be constructed using a tensor product of two such chain complexes. Here we will investigate the "three-dimensional" version of that construction.

Consider three classical error-correcting codes with parity check matrices A, B, and C. The three-fold tensor product of the associated chain complexes is

$$T = C_A \otimes C_B \otimes C_C : 0 \xrightarrow{\partial_4} T_3 \xrightarrow{\partial_3} T_2 \xrightarrow{\partial_2} T_1 \xrightarrow{\partial_1} T_0 \xrightarrow{\partial_0} 0. \tag{4}$$

The vector spaces are

$$T_0 = A_0 \otimes B_0 \otimes C_0, \tag{5}$$
$$T_1 = A_1 \otimes B_0 \otimes C_0 \oplus A_0 \otimes B_1 \otimes C_0 \oplus A_0 \otimes B_0 \otimes C_1, \tag{6}$$
$$T_2 = A_1 \otimes B_1 \otimes C_0 \oplus A_0 \otimes B_1 \otimes C_1 \oplus A_1 \otimes B_0 \otimes C_1, \tag{7}$$
$$T_3 = A_1 \otimes B_1 \otimes C_1. \tag{8}$$

The action of each boundary operator is defined by the Leibniz product rule, for example:

$$\partial_1 = \begin{pmatrix} \partial_{A_1} \otimes I_{B_0} \otimes I_{C_0} & I_{A_0} \otimes \partial_{B_1} \otimes I_{C_0} & I_{A_0} \otimes I_{B_0} \otimes \partial_{C_1} \end{pmatrix}, \tag{9}$$

where the three components of this row vector act on the three summands of T_1 (and we have used $\partial_0 A_0 = 0$). We will regard ∂_1 as the H_X check matrix for the X-type stabilizer generators of a CSS quantum code. Then we can choose ∂_2^T as the H_Z check matrix for the Z-type stabilizer generators, and the CSS condition $H_X H_Z^T$ will follow automatically from the property $\partial_1 \circ \partial_2 = 0$ of the chain complex. We can think of H_X as acting on three different qubit "registers" corresponding to the three summands in T_1.

a) The boundary operator ∂_2 is also defined by the Leibniz product rule, and its action on the three registers can be expressed as a 3×3 matrix where each entry is a tensor product of three matrices. Write out this matrix in terms of ∂_{A_1}, ∂_{B_1}, ∂_{C_1}, I_{A_0}, I_{B_0}, I_{C_0}, I_{A_1}, I_{B_1}, I_{C_1}, and verify $\partial_1 \circ \partial_2 = 0$ (over the binary field $\mathbb{F}_2$).

b) Write the H_X and H_Z parity check matrices for the corresponding CSS code in terms of the classical code parity check matrices, A, B, C, their transposes, and identity matrices. Note that the Z-type check operators come in three varieties. Verify that all X-type stabilizers commute with all Z-type stabilizers.

c) Express the number n of physical qubits in this code block in terms of $n_{A,B,C}$ and $r_{A,B,C}$.

The Künneth formula for the homology of a tensor product of three chain complexes says

$$H_n(C_A \otimes C_B \otimes C_C) = \bigoplus_{i+j+k=n} H_i(C_A) \otimes H_j(C_B) \otimes H_k(C_C). \tag{10}$$

d) Using this formula, express the number k of encoded qubits in the CSS code in terms of the dimensions $k_{A,B,C}$ of the classical codes and the dimensions $k^T_{A,B,C}$ of their transpose codes.

e) Using this same tensor product complex, an alternative CSS code can be constructed by choosing H_X to be ∂_2, and H_Z to be ∂_3^T. Find n and k for this alternative code.

2.2 The three-dimensional and four-dimensional toric codes

Recall that the $L \times L$ toric code is the hypergraph product of two distance-L classical repetition codes. These classical codes are defined on a cycle — that is, the parity check matrix H is $L \times L$, with one redundant row.

The three-dimensional $L \times L \times L$ toric code is similarly defined as a product of three length-L classical repetition codes. As noted in the previous exercise, there are two versions of the code, one in which $H_X = \partial_1$, $H_Z = \partial_2^T$ and one in which $H_X = \partial_2$, $H_Z = \partial_3^T$.

a) For each version of the code, find n and k.

b) In addition, find for each version the distances d_X (smallest weight of a nontrivial X-type logical operator) and d_Z (smallest weight of a Z-type logical operator).

c) The four-dimensional toric code is the product of four length-L repetition codes. Describe the vector spaces and boundary operators of the associated chain complex.

d) For the 4D CSS code with $H_X = \partial_2$ and $H_Z = \partial_3^T$, find n, k, d_X, and d_Z.

e) For the 4D CSS code with $H_X = \partial_1$ and $H_Z = \partial_2^T$, find n, k, d_X, and d_Z.

f) A *metacheck* is a topological constraint satisfied by error syndromes. Metachecks provide protection against syndrome measurement errors, enabling "single-shot" error correction. In one version of the 4D toric code (either the one in (d) or the one in (e)), there are metachecks constraining both the X and Z syndromes. Explain how this works in the chain-complex language.

2.3 Counting qubits in a lifted product code

Consider a lifted product code over the polynomial ring $R = \mathbb{F}_2[x]/(x^L - 1)$ with seed matrix

$$H(x) = \begin{pmatrix} 1 & x \\ x^2 & 1 \end{pmatrix}, \tag{11}$$

a 2×2 matrix over R. Hence the X and Z check matrices are

$$H_X = (H \otimes I, I \otimes H), \quad H_Z = (I \otimes H^T, H^T \otimes I), \tag{12}$$

where I is the 2×2 identity matrix over R. (Note that in taking the transpose the circulant matrix x is mapped to its transpose x^{-1}.) Regarding x and x^2 as circulant $L \times L$ matrices, H_X and H_Z are both $4L \times 8L$. Therefore this CSS code has $n = 8L$ physical qubits, $4L$ X-type stabilizer generators and $4L$ Z-type stabilizer generators.

a) Show that the number k of encoded qubits is $k = 2q^2$ where $q = \dim\ker H(x) = \dim\ker H^T(x)$. (Use what you know about hypergraph product codes.)

b) Show that the nullity q is equal to the number of Lth roots of unity ω (counted with multiplicity) for which $\det H(\omega) = 0$. (Circulant matrices can be diagonalized by Fourier transforming.)

c) Compute k assuming that L is odd. How does the answer depend on whether or not 3 divides L?

2.4 Parameters of a bivariate bicyle code

As discussed in class (and in the notes), the toric code on an $L \times L$ lattice can be regarded as a bivariate bicycle (BB) code with

$$H_X = \begin{pmatrix} A(x,y) & B(x,y) \end{pmatrix}, \quad H_Z = \begin{pmatrix} B(x^{-1},y^{-1}) & A(x^{-1},y^{-1}) \end{pmatrix}, \tag{13}$$

where

$$A = 1 + x, \quad B = 1 + y \tag{14}$$

and $x^L = y^L = 1$. Consider instead the case

$$A = 1 + x^a, \quad B = 1 + y^b. \tag{15}$$

a) Show that the number k of encoded qubits is $k = 2q$, where $q = \dim \ker(A, B)$.

b) Show that q is equal to the number of pairs (ω_x, ω_y) such that

$$\omega_x^L = \omega_y^L = 1, \quad A(\omega_x, \omega_y) = 0, \quad B(\omega_x, \omega_y) = 0. \tag{16}$$

c) Assume L is odd. Solve these equations and show that $q = g_a g_b$ where g_a is the greatest common divisor of a and L, and g_b is the greatest common divisor of b and L. Hence determine k.

d) Argue that the code distance is $d = \min(L/g_a, L/g_b)$.