Ph/CS 219C

Exercises
Due: Monday 8 June 2026

4.1 The folded surface code

Consider the [[13,1,3]] unrotated surface code, with a line drawn across the diagonal as shown in Fig. 1. A pair of qubits at positions that are located off the diagonal and related by reflection across the diagonal may be called a "mirror pair." We can imagine folding the code block at the diagonal, obtaining a triangular shape with two layers, such that qubits in a mirror pair sit on top or one another. In this folded geometry, a two-qubit operation acting on a mirror pair becomes geometrically local. Even if the fold is merely conceptual rather than physically realized, in an quantum processor with nonlocal connectivity there is no obstacle to performing operations on a mirror pair.

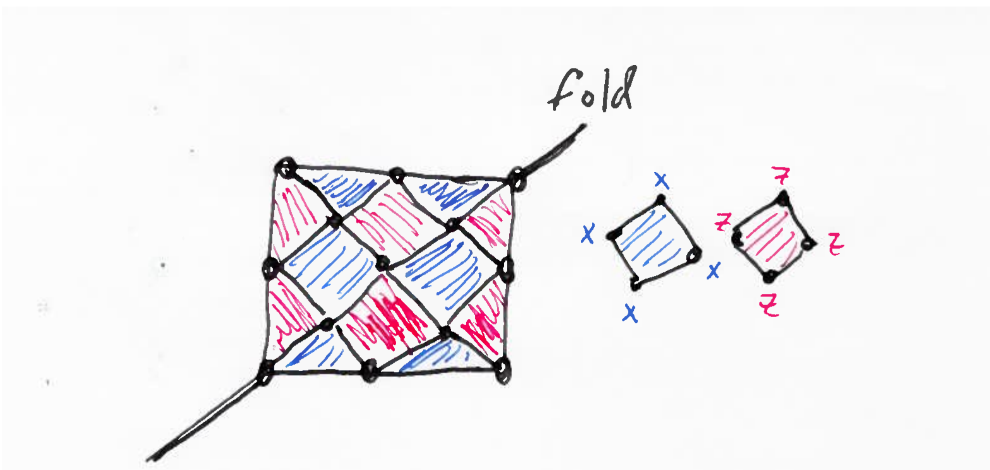

Figure 1: Folded surface code.

a) Suppose that a Hadamard gate is applied to each of 13 qubits, after which the two qubits in each mirror pair swap positions. Show that this procedure preserves the code space.

b) What logical operation acting on the code space is achieved through this procedure?

c) Now suppose that S and $S^\dagger$ gates are applied on the diagonal, in the sequence $S \otimes S^\dagger \otimes S \otimes S^\dagger \otimes S$ running along the diagonal from the upper right corner to the lower left corner, after which a CZ gates is applied to each mirror pair. Show that this procedure preserves the code space. (The X-type check operators are transformed to operators that are not X-type, but you should ask whether the resulting operators can be transformed back to X-type checks by applying Z-type checks.)

d) What logical operation acting on the code space is achieved through this procedure?

e) What happens if we apply this procedure to the $d \times d$ surface code for $d > 3$?

4.2 From short-range to long-range entanglement via measurement and feed forward

Consider a family of spatial lattices with a qubit occupying each lattice site. We say a quantum state of these qubits is short-range entangled if, starting from a product state, it can be prepared by a geometrically local unitary quantum circuit with depth (number of time steps) that does not depend on the system size. We have seen that cluster states (or more generally graph states) are short-range entangled, as they can be created, starting with a product of $|+\rangle$ states, by CZ gates acting on each qubit pair connected by a lattice edge. The surface code, on the other hand, is long-range entangled. In this problem we'll see how long-range entangled states can be obtain from cluster states by performing measurements of Pauli operators on a subset of the qubits and then updating the Pauli frame.

a) Consider a 1D cluster state with an odd number of sites. Suppose that all the qubits are measured in the X basis except for the two qubits at the ends of the chain on the left and the right. The resulting two-qubit state is the simultaneous eigenstate of two commuting Pauli operators. What are these Pauli operators? How do their eigenvalues depend on the measurement outcomes? What two-qubit state is this? (You can find stabilizers for the 1D cluster state such that the only Pauli operators acting on the interior of the chain are X and I.)

b) What if the length of the chain is even rather than odd?

c) Now consider the 2D cluster state on the graph shown in Fig. 2. This is a square lattice with dangling edges at the top and bottom, and a qubit resides on each site and on each edge. There are 6 site qubits and 13 edge qubits. Suppose that all the site qubits are measured in the X basis. The result is a 13-qubit code state for a CSS code with geometrically local X-type and Z-type check operators. Indicate these check operators, distinguishing the X-type and Z-type checks by coloring cells of the resulting 13-qubit lattice. What is this CSS code? Express the eigenvalues of the check operators in terms of the measurement outcomes. (You can find geometrically local stabilizers for the 2D cluster state such that the only Pauli operators acting on site-qubits are X and I.)

d) Because the cluster state with which we began is a unique state (a one-dimensional code), this procedure produces a unique state of the resulting CSS code. What state is it?

e) How does this procedure generalize to larger 2D clusters states with dangling edges on the top and bottom?

4.3 A stabilizer code and a subsystem code

Figure 2: A lattice with hanging edges at the top and bottom.

a) Consider an $n = 4$ CSS code and imagine the four physical qubits are arranged in a 2×2 grid. Suppose the code's logical Pauli operators are

$$X_1 = \begin{pmatrix} X & X \\ I & I \end{pmatrix}, \quad Z_1 = \begin{pmatrix} Z & I \\ Z & I \end{pmatrix}, \quad X_2 = \begin{pmatrix} X & I \\ X & I \end{pmatrix}, \quad Z_2 = \begin{pmatrix} Z & Z \\ I & I \end{pmatrix}. \tag{1}$$

What is the center of the group generated by these four operators?

b) What is the code's stabilizer?

c) Explain how to do a logical CNOT on these two encoded qubits by permuting physical qubits. (The action of CNOT by conjugation is $X_1 \to X_1X_2$, $X_2 \to X_2$, $Z_1 \to Z_1$, $Z_2 \to Z_1Z_2$.)

d) Now consider an $n = 4$ code with the gauge generators

$$X_1 = \begin{pmatrix} X & X \\ I & I \end{pmatrix}, \quad Z_1 = \begin{pmatrix} Z & I \\ Z & I \end{pmatrix}, \quad X_2 = \begin{pmatrix} I & I \\ X & X \end{pmatrix}, \quad Z_2 = \begin{pmatrix} I & Z \\ I & Z \end{pmatrix}. \tag{2}$$

What is the center of the gauge group and what is the code's stabilizer?

e) Explain how to determine the eigenvalues of the stabilizer generators by measuring weight-2 Pauli operators. Is this possible for the code described in (*a*)?

f) What logical qubit is protected by this code?